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ON FUZZY UPPER AND LOWER WEAKLY e^*-CONTINUOUS MULTIFUNCTIONS

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Abstract

In this paper, we introduce the concepts of fuzzy upper and fuzzy lower weakly e^*-continuous multifunction on fuzzy topological spaces in $\hat{S}$ostak sense. Several characterizations and properties of fuzzy upper (resp. lower) weakly e^*-continuous multifunctions are presented and their mutual relationships are established in L-fuzzy topological spaces.

Key Words and Phrases : *Fuzzy upper (resp. lower) weakly e^*-continuous multifunction, Fuzzy upper (resp. lower) almost e^*-continuous multifunction.*

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1. Introduction

Kubiak [16] and $\hat{S}$ostak [24] introduced the notion of (L-)fuzzy topological space as a generalization of L-topological spaces (originally called (L-) fuzzy topological spaces by Chang [6] and Goguen [9]. It is the grade of openness of an L-fuzzy set. A general approach to the study of topological type structures on fuzzy powersets was developed in ([10-12], [16], [17], [24-26]).

Berge [5] introduced the concept multimapping $F : X \multimap Y$ where X and Y are topological spaces and Popa [22,23] introduced the notion of irresolute multimapping. After Chang introduced the concept of fuzzy topology [6], continuity of multifunctions in fuzzy topological spaces have been defined and studied by many authors from different view points (eg. see [3], [4], [19-21]). Tsiporkova et al.,[30,31] introduced the continuity of fuzzy multivalued mappings in the Chang's fuzzy topology [6]. Later, Abbas et al., [1] introduced the concepts of fuzzy upper and fuzzy lower semi-continuous multifunctions in L-fuzzy topological spaces. Recently, Sobana et al. [29] and Vadivel et al. [34] introduced the concept of r-fuzzy e and e^*-open sets and r-fuzzy e and e^*-continuity in $\check{S}$ostak's fuzzy topological spaces.

In this paper, we introduce the concepts of fuzzy upper and fuzzy lower weakly e^*-continuous multifunction on fuzzy topological spaces in $\hat{S}$ostak sense. Several characterizations and properties of these multifunctions are presented and their mutual relationships are established in L-fuzzy topological spaces.

Throughout this paper, nonempty sets will be denoted by X, Y etc., $L = [0,\ 1]$ and $L_0 = (0,\ 1]$. The family of all fuzzy sets in X is denoted by L^X. The complement of an L-fuzzy set λ is denoted by λ^c. This symbol $\multimap$ for a multifunction.

For $\alpha \in L$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. A fuzzy point x_t for $t \in L_0$ is an element of L^X such that $x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$ The family of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ iff $t \leq \lambda(x)$.

All other notations are standard notations of L-fuzzy set theory.

2. Preliminaries

Definition 2.1 [1] : Let $F : X \multimap Y$, then F is called a fuzzy multifunction (FM, for short) if and only if $F(x) \in L^Y$ for each $x \in X$. The degree of membership of y in $F(x)$

is denoted by $F(x)(y) = G_F(x,\ y)$ for any $(x,\ y) \in X \times Y$. The domain of F, denoted by $domain(F)$ and the range of F, denoted by $rng(F)$, for any $x \in X$ and $y \in Y$, are defined by :

$$dom(F)(x) = \bigvee_{y \in Y} G_F(x,\ y) \ \text{ and } \ rng(F)(y) = \bigvee_{x \in X} G_F(x,\ y).$$

Definition 2.2 [**1**] : Let $F : X \multimap Y$ be a FM. Then F is called:

(i) Normalized iff for each $x \in X$, there exixts $y_0 \in Y$ such that $G_F(x,\ y_0) = \overline{1}$.

(ii) A crisp iff $G_F(x,\ y) = \overline{1}$ for each $x \in X$ and $y \in Y$.

Definition 2.3 [**1**] : Let $F : X \multimap Y$ be a FM. Then

(i) The image of $\lambda \in L^X$ is an L-fuzzy set $F(\lambda) \in L^Y$ defined by

$$F(\lambda)(y) = \bigvee_{x \in X} [G_F(x,\ y) \wedge \lambda(x)].$$

(ii) The lower inverse of $\mu \in L^Y$ is an L-fuzzy set $F^l(\mu) \in L^X$ defined by

$$F^l(\mu)(x) = \bigvee_{y \in Y} [G_F(x,\ y) \wedge \mu(y)].$$

(iii) The upper inverse of $\mu \in L^Y$ is an L-fuzzy set $F^u(\mu) \in L^X$ defined by

$$F^u(\mu)(x) = \bigwedge_{y \in Y} [G^c_F(x,\ y) \vee \mu(y)].$$

Theorem 2.1 [**1**] : Let $F : X \multimap Y$ be a FM. Then

(i) $F(\lambda_1) \leq F(\lambda_2)$ if $\lambda_1 \leq \lambda_2$.

(ii) $F^l(\mu_1) \leq F^l(\mu_2)$ and $F^u(\mu_1) \leq F^u(\mu_2)$ if $\mu_1 \leq \mu_2$.

(iii) $F^l(\mu^c) = (F^u(\mu))^c$.

(iv) $F^u(\mu^c) = (F^l(\mu))^c$.

(v) $F(F^u(\mu)) \leq \mu$ if F is a crisp.

(vi) $F^u(F(\lambda)) \geq \lambda$ if F is a crisp.

Definition 2.4 [1] : Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM. Then the composition $H \circ F$ is defined by

$$((H \circ F)(x))(z) = \bigvee_{y \in Y} [G_F(x,\ y) \wedge G_H(y,\ z)].$$

Theorem 2.2 [1] : Let $F : X \multimap Y$ and $H : Y \multimap Z$ be FM. Then we have the following

(i) $(H \circ F) = F(H)$.

(ii) $(H \circ F)^u = F^u(H^u)$.

(iii) $(H \circ F)^l = F^l(H^l)$.

Theorem 2.3 [1] : Let $F_i : X \multimap Y$ be a FM. Then we have the following

(i) $(\bigcup_{i \in \Gamma} F_i)(\lambda) = \bigvee_{i \in \Gamma} F_i(\lambda)$.

(ii) $(\bigcup_{i \in \Gamma} F_i)^l(\mu) = \bigvee_{i \in \Gamma} F_i^l(\mu)$.

(iii) $(\bigcup_{i \in \Gamma} F_i)^u(\mu) = \bigwedge_{i \in \Gamma} F_i^u(\mu)$.

Definition 2.5 [12, 16, 18, 24] : An L-fuzzy topological space (L-fts, in short) is a pair $(X,\ \tau)$, where X is a nonempty set and $\tau : L^X \to L$ is a mapping satisfying the following properties.

(1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,

(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1,\ \mu_2 \in I^X$.

(3) $\tau(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} \tau(\mu_i)$, for any $\{\mu_i\}_{i \in \Gamma} \subset I^X$,

Then τ is called an L-fuzzy topology on X. For every $\lambda \in L^X$, $\tau(\lambda)$ is called the degree of openness of the L-fuzzy set λ.

A mapping $f : (X,\ \tau) \to (Y,\ \eta)$ is said to be continuous with respect to L-fuzzy topologies τ and η iff $\tau(f^{-1}(\mu)) \geq \eta(\mu)$ for each $\mu \in L^Y$.

Theorem 2.4 [7, 14, 15, 18] : Let $(X,\ \tau)$ be a an L-fts. Then for each $\lambda \in L^X,\ r \in L_0$, we define L-fuzzy operators C_τ and $I_\tau : L^X \times L_0 \to L^X$ as follows:

$C_\tau(\lambda,\ r) = \bigwedge\{\mu \in L^X : \lambda \leq \mu,\ \tau(\overline{1} - \mu) \geq r\}$.

$I_\tau(\lambda,\ r) = \bigvee\{\mu \in L^X : \lambda \geq \mu,\ \tau(\mu) \geq r\}$.

For $\lambda,\ \mu \in L^X$ and $r,\ s \in L_0$, the operator C_τ satisfies the following conditions:

(1) $C_\tau(\overline{0}, r) = \overline{0}$,

(2) $\lambda \leq C_\tau(\lambda,\ r)$,

(3) $C_\tau(\lambda,\ r) \vee C_\tau(\mu,\ r) = C_\tau(\lambda \vee \mu,\ r)$,

(4) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$,

(5) $C_\tau(\lambda,\ r) = \lambda$ iff $\tau(\lambda^c) \geq r$.

(6) $C_\tau(\lambda^c,\ r) = (I_\tau(\lambda,\ r))^c$ and $I_\tau(\lambda^c,\ r) = (C_\tau(\lambda,\ r))^c$.

Definition 2.6 [1] : Let $F : X \multimap Y$ be a FM between two L-fts's $(X,\ \tau)$, $(Y,\ \eta)$ and $r \in L_0$. Then F is called:

(i) Fuzzy upper semi (or Fuzzy upper) (in short, FUS (or FU))-continuous at a L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(\mu)$. F is FU-continuous iff it is FU-continuous at every $x_t \in dom(F)$.

(ii) Fuzzy lower semi (or Fuzzy lower) (in short, FLS (or FL))-continuous at a L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$. F is FL-continuous iff it is FL-continuous at every $x_t \in dom(F)$.

(iii) Fuzzy continuous if it is FU-continuous and FL-continuous.

Theorem 2.5 [1] : Let $F : X \multimap Y$ be a fuzzy multifunction between two L-fts's $(X,\ \tau)$ and $(Y,\ \eta)$. Let $\mu \in L^Y$. Then we have the following

(1) F is FL-continuous iff $\tau(F^l(\mu)) \geq \eta(\mu)$.

(2) If F is normlized, then F is FU-continuous iff $\tau(F^u(\mu)) \geq \eta(\mu)$.

(3) F is FL-continuous iff $\tau(\overline{1} - F^u(\mu)) \geq \eta(\overline{1} - \mu)$.

(4) If F is normalized, then F is FU-continuous iff $\tau(\overline{1} - F^l(\mu)) \geq \eta(\overline{1} - \mu)$.

Definition 2.7 [13] : Let $(X,\ \tau)$ be a fts. For $\lambda,\ \mu \in I^X$ and $r \in I_0$, λ is called r-fuzzy regular open (for short, r-fro) (resp. r-fuzzy regular closed (for short, r-frc)) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$ (resp. $\lambda = C_\tau(I_\tau(\lambda, r), r)$).

Definition 2.8 [13] : Let $(X,\ \tau)$ be a fts. Then for each $\mu \in I^X,\ x_t \in P_t(X)$ and $r \in I_0$,

(i) μ is called r-open Q_τ-neighbourhood of x_t if $x_t q \mu$ with $\tau(\mu) \geq r$.

(ii) μ is called r-open R_τ-neighbourhood of x_t if $x_t q \mu$ with $\mu = I_\tau(C_\tau(\mu,\ r),\ r)$.

We denoted

$$Q_\tau(x_t,\ r) = \{\mu \in I^X : x_t q \mu,\ \tau(\mu) \geq r\},$$

$$R_\tau(x_t,\ r) = \{\mu \in I^X : x_t q \mu,\ \mu = I_\tau(C_\tau(\mu,\ r),\ r)\}.$$

Definition 2.9 [13] : Let $(X,\ \tau)$ be a fts. Then for each $\lambda \in I^X,\ x_t \in P_t(X)$ and $r \in I_0$,

(i) x_t is called r-τ cluster point of λ if for every $\mu \in Q_\tau(x_t,\ r)$, we have $\mu q \lambda$.

(ii) x_t is called r-δ cluster point of λ if for every $\mu \in R_\tau(x_t,\ r)$, we have $\mu q \lambda$.

(iii) An δ-closure operator is a mapping $D_\tau : I^X \times I \to I^X$ defined as follows: $\delta C_\tau(\lambda,\ r)$ or $D_\tau(\lambda,\ r) = \bigvee\{x_t \in P_t(X) : x_t$ is r-δ-cluster point of $\lambda\}$.
Equivalently, $\delta C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda,\ \mu$ is a r-frc set$\}$ and $\delta I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \mu$ is a r-fro set$\}$.

Definition 2.10 [13] : Let $(X,\ \tau)$ be a fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, λ is called r-fuzzy δ-closed iff $\lambda = \delta C_\tau(\lambda,\ r)$ or $D_\tau(\lambda,\ r)$.

Definition 2.11 [2] : Let $F : X \multimap Y$ be a FM between two L-fts's $(X,\ \tau),\ (Y,\ \eta)$ and $r \in L_0$. Then F is called:

(i) Fuzzy upper almost continuous (FUA-continuous, in short) at any L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(I_\eta(C_\eta(\mu, r), r))$.

(ii) Fuzzy lower almost continuous (FLA-continuous, in short) at any L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(I_\eta(C_\eta(\mu, r), r))$.

(iii) FUA-continuous (resp. FLA-continuous) iff it is FUA-continuous (resp. FLA-continuous) at every $x_t \in dom(F)$.

Definition 2.12 [2] : Let $F : X \multimap Y$ be a FM between L-fts's $(X,\ \tau)$, $(Y,\ \eta)$ and $r \in L_0$. Then F is called.

(i) Fuzzy upper weakly continuous (FUW-continuous, for short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(C_\eta(\mu,\ r))$

(ii) Fuzzy lower weakly continuous (FLW-continuous, for short)) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^l(C_\eta(\mu,\ r))$

(iii) FUW-continuous (resp. FLW-continuous) iff it is FUW-continuous (resp. FLW-continuous) at every $x_t \in dom(F)$.

Definition 2.13 [29] : Let $(X,\ \tau)$ be a an L-fts. Then for each $\lambda,\ \mu \in L^X,\ r \in L_0$. Then λ is called

(1) λ is called an r-fuzzy e-open (briefly, r-feo) set if $\lambda \leq C_\tau(\delta_\tau(\lambda,\ r),\ r) \vee I_\tau(\delta C_\tau(\lambda,\ r),\ r)$.

(2) λ is called an r-fuzzy e-closed (briefly, r-feo) set if $C_\tau(\delta I_\tau(\lambda,\ r),\ r) \wedge I_\tau(\delta C_\tau(\lambda,\ r),\ r) \leq \lambda$.

Definition 2.14 [29] : Let $(X,\ \tau)$ be an L-fts. Then for each $\lambda,\ \mu \in L^X,\ r \in L_0$. Then λ is called

(i) $eI_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \mu$ is a r-feo set $\}$ is called the r-fuzzy e-interior of λ.

(ii) $eC_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda,\ \mu$ is a r-fec set $\}$ is called the r-fuzzy e-closure of λ.

Definition 2.15 [27] : Let $F : X \multimap Y$ be a FM between two L-fts's $(X,\ \tau)$, $(Y,\ \eta)$ and $r \in L_0$. Then F is called:

(i) Fuzzy upper almost e^*-continuous ($FUAe^*$-continuous, in short) at any L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exist r-fe*o set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(I_\eta(C_\eta(\mu, r), r))$.

(ii) Fuzzy lower almost e^*-continuous ($FLAe$-continuous, in short) at any L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exist r-fe*o set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(I_\eta(C_\eta(\mu, r), r))$.

(iii) $FUAe^*$ -continuous (resp. $FLAe^*$-continuous) iff it is $FUAe^*$-continuous (resp. $FLAe^*$-continuous) at every $x_t \in dom(F)$.

3. Fuzzy Upper and Lower Weakly e^*-continuous Multifunctions

Definition 3.1 : Let $F : X \multimap Y$ be a FM between two L-fts's $(X,\ \tau)$, $(Y,\ \eta)$ and $r \in L_0$. Then F is called.

(i) Fuzzy upper weakly e^*-continuous ($FUWe^*$-continuous, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exist r-fe*o-set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(C_\eta(\mu,\ r))$.

(ii) Fuzzy lower weakly e^*-continuous ($FLWe^*$-continuous, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exist r-fe*o-set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(C_\eta(\mu,\ r))$.

(iii) $FUWe^*$-continuous (resp. $FLWe^*$-continuous) iff it is $FUWe^*$-continuous (resp. $FLWe^*$-continuous) at every $x_t \in dom(F)$.

Proposition 3.1 : If F is normalized, then F is $FUWe^*$-continuous at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exists r-fe*o-set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^u(C_\eta(\mu,\ r))$.

Theorem 3.1 : Let $F : X \multimap Y$ be a FM between two L-fts's $(X,\ \tau)$ and $(Y,\ \eta)$. Then F is $FLWe^*$-continuous if and only if $F^l(\mu) \leq e^*I_\tau(F^l(C_\eta(\mu,\ r)),\ r)$ for any $\mu \in L^Y$ and $\eta(\mu) \geq r$.

Proof : Let F be $FLWe^*$-continuous, $\mu \in L^Y$ and $\eta(\mu) \geq r$. If $x_t \in F^l(\mu)$, then there exixts r-fe*o set $\lambda \in L^X$ and $x_t \in \lambda$ such that $\lambda \leq F^l(C_\eta(\mu,\ r))$ and hence $\lambda \leq e^*I_\tau(F^l(C_\eta(\mu,\ r)),\ r)$. Thus $F^l(\mu) \leq e^*I_\tau(C_\eta(\mu,\ r),\ r)$.

Conversely, let $x_t \in dom(F)$, $\mu \in L^Y$, $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Then

$$x_t \in F^l(\mu) \leq e^*I_\eta(F^l(C_\eta(\mu,\ r)),\ r) = \lambda\ (say).$$

Thus, $x_t \in \lambda$ and λ is r-fe*o set such that

$$\lambda = e^* I_\tau(F^l(C_\eta(\mu,\ r)),\ r) \leq F^l(C_\eta(\mu,\ r)).$$

Hence, F is $FLWe^*$-continuous. □

Theorem 3.2 : Let $F : X \multimap Y$ be a FM and normalized between two L-fts's $(X,\ \tau), (Y,\ \eta)$. Then F is $FUWe^*$-continuous if and only if $F^u(\mu) \leq e^* I_\tau(F^u(C_\eta(\mu,\ r)),\ r)$ for any $\mu \in L^Y$ and $\eta(\mu) \geq r$..

Proof : This can be proved in a similar way as the above Theorem 3.1 □

Remark 3.1 : The following implications hold.

(i) FUW-continuous $\Rightarrow$ $FUWe^*$-continuous.

(ii) FLW-continuous $\Rightarrow$ $FLWe^*$-continuous.

The converse of the above Remark 3.1 need not be true as shown by the following examples.

Example 3.1 : Let $X = \{x_1,\ x_2\}$, $Y = \{y_1,\ y_2,\ y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1,\ y_1) = 0.8$, $G_F(x_1,\ y_2) = 0.9$, $G_F(x_1,\ y_3) = 0.8$, $G_F(x_2,\ y_1) = \overline{1}$, $G_F(x_2,\ y_2) = 0.7$, and $G_F(x_2,\ y_3) = 0.9$. Let λ_1 and λ_2 be a fuzzy subsets of X be defined as follows: $\lambda_1(x_1) = 0.3$, $\lambda_1(x_2) = 0.1$; $\lambda_2(x_1) = 0.7$, $\lambda_2(x_2) = 0.7$ and μ be a fuzzy subset of Y defined as $\mu(y_1) = 0.3$, $\mu(y_2) = 0.1$, $\mu(y_3) = 0.2$. We assume that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \mu = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

are fuzzy topologies on X and Y. For $r = \frac{1}{2}$, then F is $FUWe^*$-continuous but not FUW-continuous because μ is $\frac{1}{2}$-fuzzy open set in Y, $F^u(C_\eta(\mu, r)) = \lambda_2$ is not $\frac{1}{2}$-fuzzy open set in X.

Example 3.2 : Let $X = \{x_1,\ x_2\}$, $Y = \{y_1,\ y_2,\ y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1,\ y_1) = 0.2$, $G_F(x_1,\ y_2) = \overline{1}$, $G_F(x_1,\ y_3) = \overline{0}$, $G_F(x_2,\ y_1) = 0.5$, $G_F(x_2,\ y_2) = \overline{0}$, and $G_F(x_2,\ y_3) = 0.3$. Let λ_1 and λ_2 be a fuzzy subsets of X be defined as follows: $\lambda_1(x_1) = 0.4$, $\lambda_1(x_2) = 0.3$; $\lambda_2(x_1) = 0.9$, $\lambda_2(x_2) = 0.5$ and μ be a fuzzy subset of Y defined as $\mu(y_1) = 0.4$, $\mu(y_2) = 0.1$, $\mu(y_3) = 0.1$. We assume that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \mu = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

are fuzzy topologies on X and Y. For $r = \frac{1}{2}$, then F is $FLWe^*$-continuous but not FLW-continuous because μ is $\frac{1}{2}$-fuzzy open set in Y, $F^l(C_\eta(\mu, r)) = \lambda_2$ is not $\frac{1}{2}$-fuzzy open set in X.

Remark 3.2 : The following implications hold.

(i) FUA-continuous $\Rightarrow$ $FUAe^*$-continuous.

(ii) FLA-continuous $\Rightarrow$ $FLAe^*$-continuous.

The converse of the above Remark **??** need not be true as shown by the following example.

Example 3.3 : Let $X = \{x_1,\ x_2\}$, $Y = \{y_1,\ y_2,\ y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1,\ y_1) = 0.1$, $G_F(x_1,\ y_2) = \overline{1}$, $G_F(x_1,\ y_3) = 0$, $G_F(x_2,\ y_1) = 0.5$, $G_F(x_2,\ y_2) = \overline{0}$, and $G_F(x_2,\ y_3) = \overline{1}$. Let λ_1 and λ_2 be a fuzzy subsets of X be defined as $\lambda_1(x_1) = 0.3$, $\lambda_1(x_2) = 0.5$; $\lambda_2(x_1) = 0.5$, $\lambda_2(x_2) = 0.5$, μ_1 and μ_2 be a fuzzy subsets of Y defined as $\mu_1(y_1) = 0.5$, $\mu_1(y_2) = 0.5$, $\mu_1(y_3) = 0.5$ and $\mu_2(y_1) = 0.4$, $\mu_2(y_2) = 0.4$, $\mu_2(y_3) = 0.4$. We assume that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \mu = \mu_1,\ \mu_2, \\ 0, & \text{otherwise.} \end{cases}$$

are fuzzy topologies on X and Y. For $r = \frac{1}{2}$, then F is
(i) $FUAe$-continuous but not FUA-continuous because μ_1 is $\frac{1}{2}$-fro set in Y, $F^u(\mu_1) = \lambda_2$ is not $\frac{1}{2}$-fuzzy open set in X.
(ii) $FLAe$-continuous but not FLA-continuous because μ_1 is $\frac{1}{2}$-fro set in Y, $F^l(\mu_1) = \lambda_2$ is not $\frac{1}{2}$-fuzzy open set in X.

Theorem 3.3 : Let $\{F_i\}_{i\in\Gamma}$ be a family of $FLAe^*$-continuous between two L-fts's $(X,\ \tau)$ and $(Y,\ \eta)$. Then $\bigcup_{i\in\Gamma} F_i$ is $FLAe^*$-continuous.

Proof : Let $\mu \in L^Y$, then

$$(\bigcup_{i\in\Gamma} F_i)^l(\mu) = \bigvee_{i\in\Gamma}(F_i^l(\mu)),$$

by Theorem 2.3 (ii). Since $\{F_i\}_{i\in\Gamma}$ is a family of $FLAe^*$-continuous between two L-fts's $(X,\ \tau)$ and $(Y,\ \eta)$, then $F_i^l(\mu)$ is r-fe*o set for any r-fro set μ and each $i \in \Gamma$. Then, we have $(\bigcup_{i\in\Gamma} F_i)^l(\mu) = \bigvee_{i\in\Gamma}(F_i^l(\mu))$ is r-fe*o set for any r-fro set μ. Hence $\bigcup_{i\in\Gamma} F_i$ is $FLAe^*$-continuous. □

Theorem 3.4 : Let F_1 and F_2 be two normalized $FUAe^*$-continuous between two L-fts's $(X,\ \tau)$ and $(Y,\ \eta)$. Then $F_1 \cup F_2$ is $FUAe^*$-continuous.

Proof : Let $\mu \in L^Y$, then $(F_1 \cup F_2)^u(\mu) = F_1^u(\mu) \wedge F_2^u(\mu)$ by Theorem 2.3(iii). Since F_1 and F_2 be two normalized $FUAe^*$-continuous between two L-fts's $(X,\ \tau)$ and $(Y,\ \eta)$, then $F_i^u(\mu)$ is r-fe*o-set for any r-fro set μ and $i \in \{1, 2\}$. Then, we have $(F_1 \cup F_2)^u(\mu) = F_1^u(\mu) \wedge F_2^u(\mu)$ is r-fe*o-set for each r-fro set μ. Hence $F_1 \cup F_2$ is $FUAe^*$-continuous. □

Theorem 3.5 : Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three L-fts's. If F is FLe^*-continuous and H is FLA-continuous, then $H \circ F$ is $FLAe^*$-continuous.

Proof : Let $\nu \in L^Z$, ν is r-fro set. Since H is FLA-continuous, then from Definition 2.11, $H^l(\nu)$ is r-fuzzy open set in Y. Also, F is FLe^*-continuous $F^l(H^l(\nu))$ is r-fe*o set in Y. Hence, we have $(H \circ F)^l(\nu) = F^l(H^l(\nu))$ is $r\text{-}fe^*o$. Thus $H \circ F$ is $FLAe^*$-continuous.
□

Theorem 3.6 : Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three L-fts's. If F and H are normalized, F is FUe^*-continuous and H is FUA-continuous, then $H \circ F$ is $FUAe^*$-continuous.

Proof : Proof is similar to the above Theorem 3.5. □

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