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SOME FIXED POINT THEOREMS UNDER CONTRACTIVE CONDITIONS IN FUZZY SYMMETRIC SPACES

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Abstract

In this paper, common fixed point theorems for self-mappings of a Fuzzy symmetric space are proved. Using weakly compatibility, property (E.A), we have generalized the notion of non-compatible maps in the setting of Fuzzy symmetric spaces.

1. Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. The study of non-compatible maps have been initiated by the Authors Pant [8], Aamri and Moutawkil [1]. In [2], the authors gave a notion of the property (E.A) which generalizes the concept of non-compatible mappings in metric spaces and they proved some common fixed point theorems for non-compatible mappings under strict contractive conditions.

Key Words : *Fuzzy symmetric space, Compatible maps, Weakly compatible maps.*

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Symmetric spaces were introduced in 1931 Wilson [12], as metric-like spaces lacking the triangle inequality. Several fixed point results in such spaces were obtained, for example, see [13,7,3]. Hicks and Rhoades [6] established some common fixed point theorems in symmetric spaces using the fact that some of the properties of metrics are not required in the proofs of certain metric theorems.

In Fuzzy metric spaces, a few concepts of mathematical analysis have been developed by George and Veeramani [4,5] and also they have been developed the fixed point theorem in fuzzy metric space [11]. In Fuzzy metric space, the notion of compatible maps under the name of asymptoticallycommuting maps was introduced in the paper[8] and then in the paper [10], the notion of weak compatibility has been studied in fuzzy metric space.Later on Pant and Pant [9] studied the common fixed points of a pair of non-compatible maps in fuzzy metric space. The main purpose of this paper is to give some common fixed point theorems for self-mappings of a Fuzzy Symmetric space under a generalized contractive condition. These selfmappings are assumed to satisfy a new property, introduced recently in [2] on a metric space,which generalize the notion of non-compatible maps in the setting of a Fuzzy Symmetric space.

2. Preliminaries

Definition 2.1 : A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a, \ \forall \ a \in [0,1]$;

(iv) $a * b \leq c * d$, whenever $a \leq c, b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.2 : The 3-tuple $(X, \mu, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and μ is a fuzzy set in $X^2 \times (0, \infty)$ which satisfying the following conditions

(i) $\mu(x, y, t) > 0$;

(ii) $\mu(x, y, t) = 1$ if and only if $x = y$;

(iii) $\mu(x, y, t) = \mu(y, x, t)$;

(iv) $\mu(x, y, s) * \mu(y, z, t) \leq \mu(x, z, s + t)$;

(v) $\mu(x, y, .) : (0, \infty) \to (0, 1]$ is continuous; for all $x, y, z \in X$ and $t, s > 0$.

Definition 2.3 : The pair (X, μ) is called a fuzzy symmetric space if X is an arbitrary non-empty set and μ is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions:

(i) $\mu(x, y, t) > 0$;

(ii) $\mu(x, y, t) = 1$ if and only if $x = y$;

(iii) $\mu(x, y, t) = \mu(y, x, t)$;

(iv) $\mu(x, y, .) : (0, \infty) \to (0, 1]$ is continuous for all $x, y \in X$ and $t > 0$.

If (X, μ) is a fuzzy symmetric space, then μ is called fuzzy symmetric for X.

Example 2.4 : Consider $X = [0, 2)$ and $\mu(x, y, t) = \frac{t}{t+e^{|x-y|}-1}$. Let $x = 1, y = \frac{1}{2}, z = 0, t = 1, s = 0$, then (iv) of Definition 2.2 is not satisfied and hence (X, μ) is fuzzy semi-metric space but a not fuzzy metric space.

Definition 2.5 : A subset S of a fuzzy symmetric space (X, μ) is said to be μ-closed if for a sequence $\{x_n\}$ in S and a point $x \in X$, $\lim_{n\to\infty} \mu(x_n, x, t) = 1 \Rightarrow x \in S$.

Definition 2.6 : Two self-mappings f and g of a fuzzy symmetric space (X, μ) are called compatible if $\lim_{n\to\infty} \mu(fgx_n.gfx_nt) = 1$ whenever $\{x_n\}$ is asequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X, where fg denotes the composition of f and g.

Definition 2.7 : Let X be a set and fg be self-mappings of X. A point x in X is called a coincidence point of f and g if and only if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

Definition 2.8 : A pair of self-mappings S and T is called weakly compatible if they commute at their coincidence points.

Definition 2.9 : We say that a pair of self-mappings S and T satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \mu(Sx_n, l, t) = \lim_{n\to\infty} \mu(TX_n, l, t) = 1$ for some $l \in X$.

Definition 2.10 : It is clear from the Definition 2.6 that a pair of self-mappings S and T of a fuzzy symmetric space (X, μ) are called non-compatible if there exists atleast one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X, but $\lim_{n\to\infty} \mu(fgx_n, gfx_n, t) \neq 1$ or does not exist.

Definition 2.11 : The mappings $A, B, S, T : X \to X$ of a fuzzy symmetric space (X, μ) satisfy a common property (E.A) if there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} \mu(Ax_n, l, t) = \lim_{n\to\infty} \mu(Sx_n, l, t) = 1$.

$\lim_{n\to\infty} \mu(By_n, l, t) = \lim_{n\to\infty} \mu(Ty_n, l, t) = 1$ for some $I \in X$.

We denote Φ by the class of continuous function $\Phi : [0, 1] \to [0, 1]$ satisfying :

(φ_1). $\Phi(l) > l$ for all $l \in [0, 1)$, $\varphi_2)\Phi(1) = 1$.

3. Axioms of Fuzzy Symmetric Spaces

We state the following axioms:

(W_3) For a sequence $\{x_n\}$ in X, $x, y \in X$, $\lim_{n\to\infty} \mu(x_n, x, t) = 1$ and $\lim_{n\to\infty} \mu(x_n, y, t) = 1$ imply $x = y$.

(W_4) For sequences $\{x_n\}, \{y_n\}$ in X, $x \in X$, $\lim_{n\to\infty} \mu(x_n, x, t) = 1$ and $\lim_{n\to\infty} \mu(y_n, x_n, t) = 1 \Rightarrow \lim_{n\to\infty} \mu(y_n, x, t) = 1$.

(H_E) For sequences $\{x_n\}, \{y_n\}$ in $X, x \in X$, $\lim_{n\to\infty} \mu(x_n, x, t) = 1$ and $\lim_{n\to\infty} \mu(y_n, x, t) = 1 \Rightarrow \lim_{n\to\infty} \mu(x_n, y_n, t) = 1$.

(C_c) For sequence $\{x_n\}$ in X and $x, y \in X$, $\lim_{n\to\infty} \mu(x_n, x, t) = 1$ implies $\lim_{n\to\infty} \mu(x_n, y, t) = \mu(x, y, t)$.

Proposition 3.1 : For axioms in symmetric space (X, μ), 0ne has

(1) $(W_4) \Rightarrow (W_3)$,

(2) $(C_c) \Rightarrow (W_3)$.

Proof : Let $\{x_n\}$ be a sequence in X and $x, y \in X$ with $\lim_{n\to\infty} \mu(x_n, x, t) = 1$ and $\lim_{n\to\infty} \mu(x_n, y, t) = 1$.

(1). By putting $y_n = y$ for each $n \in N$, we have $\lim_{n\to\infty} \mu(x_n, x, t) = \lim_{n\to\infty} \mu(x_n, y_n, t) = 1$. By (W_4) we have

$$1 = \lim_{n\to\infty} \mu(y_n, x, t) = \lim_{n\to\infty} \mu(y, x, t) \Rightarrow x = y.$$

(2). By (C_c), $\lim_{n\to\infty} \mu(x_n, x, t) = 1 \Rightarrow \lim_{n\to\infty} \mu(x, y, t) = \lim_{n\to\infty} \mu(x_n, y, t) = 1 \Rightarrow x = y$.

4. Main Results

Theorem 4.1 : Let (X, μ) be a fuzzy symmetric space that satisfies (W_3) and (H_E). Let A and B be two weakly compatible self-mappings of X such that

(1) $\mu(Ax, Ay, t) \geq \varphi(\min\{\mu(Bx, By, t), \mu(Bx, Ay, t), \mu(Ay, By, t)\})$ for all $(x, y) \in X^2$,

(2) A and B satisfy the property $(E.A)$, and

(3) $AX \subset BX$. If the range of A or B is a complete subspace of X, then A and B have a unique common fixed point.

Proof : Since A and B satisfy the property $(E.A)$, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \mu(Ax_n, l, t) = \lim_{n\to\infty} \mu(Bx_n, l, t) = 1$ for some $l \in X$.
Therefore, by $(H.E)$, we have $\lim_{n\to\infty} \mu(Ax_n, Bx_n, t) = 1$.
Suppose that BX is a complete subspace of X. Then $l = Bu$ for some $u \in X$. We claim that $Au = Bu$. Indeed, by (1), we have

$$\begin{aligned} \mu(Au, Ax_n, t) &\geq \varphi(\min\{\mu(Bu, Bx_n, t), \mu(Bu, Ax_n, t), \mu(Bx_n, Ax_n, t)\}) \\ &> \min\{\mu(Bu, Bx_n, t), \mu(Bu, Ax_n, t), \mu(Bx_n, Ax_n, t)\}. \end{aligned}$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} \mu(Au, Ax_n, t) = 1$.
Hence, by (W_3), we have $Au = Bu$. The weak compatibility of A and B implies that $ABu = BAu$ and then $AAu = ABu = BAu = BBu$. Let us show that Au is common fixed point of A and B. Suppose that $AAu \neq Au$. In view of (1), it follows

$$\begin{aligned} \mu(Au, AAu, t) &\geq \varphi(\min\{\mu(Bu, BAu, t), \mu(Bu, AAu, t), \mu(BAu, AAu, t)\}) \\ &\geq \varphi(\min\{\mu(AAu, Au, t), \mu(AAu, Au, t\})) \\ &\geq \varphi(\mu(AAu, Au, t)) \\ &\geq \mu(AAu, Au, t) \end{aligned}$$

which is a contradiction. Therefore $Au = AAu = Bau$ and Au is a common fixed point of A and B. The proof is similar when AX is assumed to be a complete subspace of X, since $AX \subset BX$.

If $Au = Bu = u$ and $Av = Bv = v$, $u \neq v$, then (1) gives

$$\begin{aligned}\mu(u,v,t) &= \mu(Au,Av,t)\\ &\geq \varphi(\min\{\mu(Bu,Bv,t),\mu(Bu,Av,t),\mu(Bv,Av,t)\})\\ &\geq \varphi(\mu(u,v,t))\\ &> \mu(u,v,t)\end{aligned}$$

which is a contradiction. $\therefore$ $u = v$ and the common fixed point is unique.
Since two non-compatible self-mappings of a fuzzy symmetric space (X,μ) satisfy the property $(E.A)$, we get the following result.

Corollary 4.1 : Let (X,μ) be a fuzzy symmetric space that satisfies (W_3) and $(H.E)$. Let A and B be two weakly non compatible self-mappings of X such that
(1) $\mu(Ax,Ay,t) \geq \varphi(\min\{\mu(Bx,By,t),\mu(Bx,Ay,t),\mu(Ay,By,t)\})$ for all $(x,y) \in X^2$, and
(2) $AX \subset BX$. If the range of A or B is a complete subspace of X, then A and B have a unique common fixed point.

Theorem 4.2 : Let (X,μ) be a fuzzy symmetric space that satisfies (W_3), (W_4) and $(H.E)$. Let A,B,T and S be self-mappings of (X,μ) such that

(1) $\mu(Ax,By,t) \geq \varphi(\min\{\mu(Sx,Ty,t),\mu(Sx,By,t),\mu(Ty,Byt)\})$ for all $(x,y) \in X^2$,

(2) (A,T) and (B,S) are weakly compatibles,

(3) (A,S) or (B,T) satisfies the property $(E.A)$, and

(4) $AX \subset TX$ and $BX \subset SX$. If the range of one of the mappings A,B,T or S is a complete subspace of X, then A,B,T and S have a unique common fixed point.

Proof : Suppose that (B,T) satisfies the property $(E.A)$. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \mu(Bx,l,t) = \lim_{n\to\infty} \mu(Tx_n,l,t) = 1$ for some $t \in X$. Since $BX \subset SX$, there exists a sequence $\{y_n\}$ such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} \mu(Sy_n,l,t) = 0$.
Let show that $\lim_{n\to\infty} \mu(Ay_n,l,t) = 0$.
Ineed , in view of (1), we have

$$\begin{aligned}\mu(Ay_n,Bx_n,t) &\geq \phi(\min\{\mu(Sy_n,Tx_n,t),\mu(Sy_n,Bx_n,t)),\mu(Tx_n,Bx_n,t)\}\\ &\geq \phi(\min\{\mu(Bx_n,Tx_n,t),0,\mu(Tx_n,Bx_n,t)\})\\ &\geq \phi(\mu(Tx_n,Bx_n,t)) > \mu(Tx_n,Bx_n,t).\end{aligned}$$

Therefore by (H.E), one has $\lim_{n\to\infty}\mu(Ay_n, Bx_n, t) = 1$.
By (W_4), we deduce that $\lim_{n\to\infty}\mu(Ay_n, l, t) = 1$. Suppose that $SX$is a complete subspace of X. Then $t = Su$ for some $u \in X$.
Subsequently, we have

$$\lim_{n\to\infty}\mu(Ay_n, Su, t) = \lim_{n\to\infty}\mu(Bx_n, Su, t) = \lim_{n\to\infty}\mu(Tx_n, Su, t) = \lim_{n\to\infty}\mu(Sy_n, Su, t) = 1.$$

Using (1), it follows

$$\mu(Au, Bx, t) \leq \varphi(\min\{\mu(Su, Tx_n, t), \mu(Su, Bx_n, t), \mu(Tx_n, Bx_n, t)\}).$$

Letting $n \to \infty$, we have $\lim_{n\to\infty}\mu(Au, Bx_n, t) = 1$.
By (W_3), we have $Au = Su$. The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$. On the other hand, since $AX \subset TX$, there exists $v \in X$ such that $Au = Tv$. We claim that $Tv = Bv$. If not, condition (1) gives

$$\begin{aligned}
\mu(Au, Bv) &\geq \varphi(\min\{\mu(Su, Tv, g), \mu(Su, Bv, t), \mu(Tv, Bv, t)\}) \\
&\geq \varphi(\min\{\mu(Au, Bv, t), \mu(Au, Bv, t)\}) \\
&\geq \varphi(\mu(Au, Bv, t)) \\
&> \mu(Au, Bv, t)
\end{aligned}$$

which is a contradiction. Here $Au = Su = Tv = Bv$. The weak compatibility of B and T implies that $BTv = TBv$ and $TTv = TBv = BTv = BBv$. Let us show that Au is a common fixed point of A, B, T and S. Suppose that $AAu \neq Au$. We have

$$\begin{aligned}
\mu(Au, AAu, t) &= \mu(AAu, Bv, t) \\
&\geq \varphi(\min\{\mu(Say, Tv, t), \mu(SAu, Bv, t), \mu(Tv, Bv, t)\}) \\
&\geq \varphi(\min\{\mu(AAu, Au, t), \mu(AAu, Au, t)\}) \\
&\geq \varphi(\mu(AAu, Au, t)) \\
&> \mu(AAu, Au, t)
\end{aligned}$$

which is a contradiction. Therefore, $Au = AAu = SAu$ and Au is a common fixed point of A and S. Similarly, we prove that Bv is a common fixed point of Band T. Since $Au = Bv$, we conclude that Au common fixed point of A, B, S and T. The proof is similar when TX is assumed to be complete subspace of X. The cases in which AX or

BX is a complete subspace of X are similar to the cases in which TX or SX respectively is complete. Since $AX \subset TX$ and $BX \subset SX$.
If $Au = Bu = Tu = Su = u$ and $Av = Bv = Tv = Sv = v$ and $u \neq v$, then (1) gives

$$\begin{aligned} \mu(u,v) &= \mu(Au, Bv, t) \geq \varphi(\min\{\mu(Su, Tv, t), \mu(Su, Bv, t), \mu(Tv, Bv, t)\}) \\ &\geq \varphi(\mu(u, v, t)) \\ &> \varphi(u, v) \end{aligned}$$

which is a contradiction. Therefore $u = v$ and the common fixed point is unique.
Corollary 4.2 : Let A, B, T and S be self-mappings of a fuzzy symmetric space (X, μ) such that

(1) $\mu(Ax, By, t) \geq \varphi(\min\{\mu(Sx, Ty, t), \mu(Sx, By, t), \mu(Ty, By, t)\}0$ for all $(x, y) \in X^2$.

(2) (A, T) and (B, S) are weakly compatibles ,

(3) (A, S) or (B, T) satisfies the property $(E.A)$, and

(4) $AX \subset TX$ and $BX \subset SX$.

If the range of one of the mappings A, B, T or S is a complete subspace of X, then A, B, T and S have a unique common fixed point.

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