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ON THE SPLIT EDGE GEODETIC NUMBER IN GRAPHS

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Abstract

An edge geodetic set $S \subseteq V$ is said to be a split edge geodetic set if the subgraph $G[V-S]$ induced by $V-S$ is disconnected. The minimum cardinality of a split edge geodetic set of G is the split edge geodetic number and is denoted by $g_{1s}(G)$. In this paper, we initiate the study of $g_{1s}(G)$ in terms of vertices, edges and different parameters. Further we determine bounds for the split edge geodetic number of cartesian product, corona.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For vertices u and v in a connected graph, the distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. An $u-v$ path of length $d(u, v)$ is called an $u-v$ geodesic. It is known that this distance is a metric on

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the vertex set $V(G)$. For any vertex u of G, the eccentricity of u is $e(u) = \max\{d(u, v) : v \in V\}$. The radius rad G and diameter diam G are defined by rad $G = \min\{e(v) : v \in V\}$ and diam $G = \max\{e(v) : v \in V\}$ respectively. A geodetic set of G is a set $S \subseteq V$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number $g(G)$ is the minimum order of its geodetic sets. This concept was introduced in [2]. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbours is complete. A friendship graph F_n is obtained by joining n copies of the cycle graph C_3 with a common vertex. An extreme vertex is a vertex v if the subgraph induced by its neighbours is complete. The cartesian product of two graphs G and H, written as $G \square H$, is the graph with vertex set $V(G) \square V(H)$, where two distinct vertices (u_1, u_2) and (v_1, v_2) being adjacent in $G \square H$ if and only if either $u_1 = v_1$ and $u_2v_2 \in E(H)$, or $u_2 = v_2$ and $u_1v_1 \in E(G)$. Let G_1 and G_2 be the graphs of order n_1 and n_2 respectively. The corona of two graphs G_1 and G_2 is the graph $G_1 \circ G_2$ obtained by taking one copy of G_1 and n_1 copies of G_2 and then joining the ith vertex of G_1 to every vertex of the ith copy of G_2. A set $S \subseteq V(G)$ is called an edge geodetic set of G if every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_1(G)$ of G is the minimum cardinality of its edge geodetic set. This concept was introduced in [4]. A geodetic set $S \subseteq V$ is said to be split geodetic set if the subgraph $G[V - S]$ induced by $V - S$ is disconnected. The minimum cardinality of a split geodetic set of G is the split geodetic number and is denoted by $g_s(G)$. This concept was introduced in [6].

2. Split Edge Geodetic Number of a Graph

We now define the Split Edge Geodetic Number of a graph.

Definition 2.1 : An edge geodetic set $S \subseteq V$ is said to be a split edge geodetic set if the subgraph $G[V - S]$ induced by $V - S$ is disconnected. The minimum cardinality of a split edge geodetic set of G is the split edge geodetic number and is denoted by $g_{1s}(G)$.

Example 2.2 : For a connected graph G given in Figure 1, $S = \{v_1, v_4, v_5, v_7\}$ is the split edge geodetic set for G so that $g_{1s}(G) = 4$.

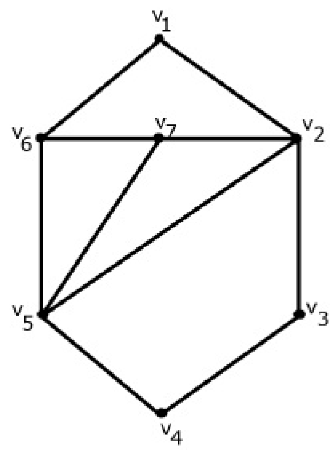

Figure 1 : G

Remark 2.3 : For a graph G given in Figure 2, $S = \{v_1, v_2, v_5\}$ is an edge geodetic set of G so that $g_1(G) = 3$. Since $V - S$ is connected, S is not split edge geodetic number. Let $S_0 = \{v_1, v_2, v_3, v_5\}$. Clearly, S_0 is split edge geodetic number of a graph so that $g_{1s}(G) = 4$. Thus edge geodetic number and split edge geodetic number of a graph are different.

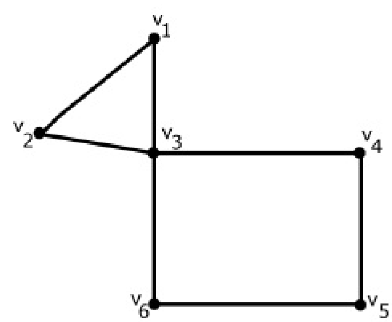

Figure 2 : G

3. Preliminary Notes

We need the following theorems to prove further results.

Theorem 3.1 [4] : For any tree T, the edge geodetic number $g_1(T)$ equals the number of end vertices in T. In fact, the set of all end vertices of T is the unique edge geodetic basis of T.

Theorem 3.2 [**4**] : For any graph G of order p, $2 \leq g_1(G) \leq p$.

Theorem 3.3 [**6**] : For any cycle C_n with $n > 3$

$$g_s[C_n] = \begin{cases} 2 & \text{if } n \text{ is even} \\ \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

Theorem 3.4 [**6**] : For any Path $P_n, n \geq 5$, $g_s[P_n] = 3$.

Theorem 3.5 [**5**] : For integers $m \geq n \geq 2$, $g_1(K_m \square K_n) = mn - n$.

The succeeding are direct results on split edge geodetic number of graphs.

Theorem 3.6 :

1. Let C_n be a cycle of order n, $n \geq 3$, then

$$g_{1s}[C_n] = \begin{cases} 2 & \text{if } n \text{ is even} \\ \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

2. For any path P_n, $g_{1s}[P_n] = 3$.

3. Each extreme vertex of a connected graph G belongs to every split edge geodetic set of G. In particular, each end vertex of G belongs to every split edge geodetic set of G.

Proof : 1. For cycle C_n proof is obvious from Theorem 3.3.

2. By Theorem 3.4, the proof is obvious.

3. By Theorem 3.1, each end vertex of a connected graph G belongs to every edge geodetic set. Therefore each extreme vertex of a connected graph G belongs to every split edge geodetic set of G.

4. Main Results

Theorem 4.1 : Let r, s be any integers and $G = K_{r,s}$, then $g_{1s}(G) = \min\{r, s\}$.

Proof : Let $A = \{a_1, a_2, \cdots, a_r\}$ and $B = \{a'_1, a'_2 \cdots, a'_s\}$ be the bipartition of G and $V = A \cup B$ where $r \leq s$. Consider $S = A$, since every vertex $a'_k, 1 \leq k \leq s$ lies on the $a_i - a_j$ geodesic for $1 \leq i \neq j \leq r$ then S is a split edge geodetic edge of G as V$-S$ is disconnected. Suppose $W = \{a_1, a_2, \cdots, a_{r-1}\}$ be the set of vertices such that $|W| < |S|$. Since $a_r \in I[W]$, then W is not a split edge geodetic set of G. Clearly S is the minimum split edge geodetic set of G. Therefore $g_{1s}(G) = \min\{r.s\}$.

Theorem 4.2 : For a complete graph K_n, a wheel W_n and a star $K_{1,s}$, there is no split edge geodetic number.

Proof : (1) Let $V(K_n) = \{u_1, u_2, \cdots, u_n\}$ be the vertex set of K_n and $d(u_i, v_j) = 1$ in K_n. Let $S = V(K_n)$. Clearly the set S of all vertices of G is an edge geodetic cover of G so that $g_1(G) = n$. Then $V - S = \phi$. Therefore there is no split edge geodetic set for K_n.

(2) Let W_n be a wheel with $V(Wn) = \{w_1, w_2, \cdots, w_n, x\}$. Let $S = \{w_1, w_2, \cdots, w_n\}$ be the edge geodetic set of W_n so that $g_1(W_n) = n-1$. Clearly $V - S$ is always connected. Hence there is no split edge geodetic set for W_n.

(3) By the Theorem 4.1 $g_{1s}(K_{1,s}) = 1$. Therefore there is no split edge geodetic set for $K_{1,s}$.

Theorem 4.3 : Let T be a tree with k end vertices and atleast three internal vertices, then $g_{1s}[T] = g_1(T) + 1$.

Proof : Since by Theorem 3.1, we have $g_1(T) = k$. Let $S = \{u_1, u_2, \cdots, u_k\}$ be the set containing end vertices of T and S itself is the minimum edge geodetic set of T. But $V - S$ is connected which is a contradiction. Let $S' = S \cup \epsilon$ where ϵ is any internal vertex of T such that it is not any supporting vertex of T. Clearly $V - S'$ is disconnected and hence S' is split edge geodetic set. Therefore $g_{1s}[T] = k + 1 = g_1(T) + 1$.

Theorem 4.4 : For a connected graph G of order n and diameter $d \geq 4$, $g_{1s}(G) \leq n - d + 2$.

Proof : Let x and y be any two vertices of G such that $d(x, y) = d$. Let P : $x = x_0, x_1, x_2, \cdots, x_d = y$ be the diametral path of length d. Let $S = V(G) - \{x_1, x_2, \cdots, x_{d-1}\}$. Then $I[S] = V(G)$ and $V - (S \cup \{v_2\})$ is disconnected. Therefore

$$\begin{aligned} g_{1s}(G) &\leq |S| + 1 \\ &= n - (d-1) + 1 \\ &= n - d + 2. \end{aligned}$$

Theorem 4.5 : Let T be a tree with atleast three internal vertices, then $2g_{1s}(T) - g_1(T) \leq m$.

Proof : Let $S = \{u_1, u_2, u_3, \cdots, u_k\}$ be the set formed by the end vertices of T which is the minimum edge geodetic set of T. Let $V' = \{v_i/2 \leq i \leq k-1\}$. Let $S' =$

$S \cup \{v_i\}$, where v_i be any internal vertex which is not support vertex. Clearly $V - S'$ is disconnected and forms a minimal split edge geodetic set of T. Then it follows that $2|S'| - |S| \leq m$. Hence $2g_{1s}(T) - g_1(T) \leq m$.

Theorem 4.6 : Let G be a connected graph of order $n \geq 5$. Then $3 \leq g_{1s}(G) \leq n-2$ and $g_{1s}(G) = 2$ if $G = C_n$ where n is even or $G = K_{2,s}$.

Proof : By Theorem 3.2 $g_1(G) \geq 2$. But for a split edge geodetic cover needs atleast three vertices and therefore $g_{1s}(G) \geq 3$. Thus $3 \leq g_{1s}(G) \leq n-2$. Further by Theorem 3.6, $g_{1s}(C_n) = 2$ when n is even and by Theorem 4.1, $g_{1s}(K_{2,s}) = 2$.

5. Cartesian Product

Theorem 5.1 : Let $G = P_{n_1}$ and $H = K_{1,n_2}$ then

$$g_{1s}(G \Box H) = \begin{cases} n_2 + 1 & \text{when } n_1 = 2, 3, 4 \text{ and } n_2 > 2 \\ \\ n_2 + 2 & \text{when } n_1 > 4 \text{ and } n_2 \geq 2 \end{cases} .$$

Proof : We shall discuss the following cases :

Case 1 : For $n_1 = 2, 3, 4$. Let $S = \{(u_1, v_1), (u_i, v_j)$ for $i = n_1$ and $2 \leq j \leq n_2 - 1$ where (u_i, v_j) is the vertex choosen from n_1 copy of K_{1,n_2} such that $d((u_1, v_1), (u_i, v_j)) = diam(G \Box H)$. Clearly S is an edge geodetic set and $V - S$ is connected. Let $S' = S \cup \{w_1\}$ where w_1 is the vertex which was the internal vertex of K_{1,n_2} and it lies in $(n_1 - 1)^{th}$ copy of K_{1,n_2} in $G \Box H$ be the set such that $V - S'$ is disconnected. Hence S' is the split edge geodetic set of $G \Box H$. Thus

$$\begin{aligned} g_{1s}(G \Box H) &= |S'| \\ &= |S| + \{w_1\} \\ &= n_2 + 1. \end{aligned}$$

Case 2 : For $n_1 > 4$ and $n_2 > 2$. Let $S = \{(u_1, v_1), (u_i, v_j)\}$ for $i = n_1$ and $2 \leq j \leq n_2 - 1$ where (u_i, v_j) is the vertex choosen from n_1 copy of K_{1,n_2} such that $d((u_1, v_1), (u_i, v_j)) = diam(G \Box H)$. Clearly S is an edge geodetic set and $V - S$ is connected. Let $S' = S \cup \{w_1\} \cup \{w_2\}$ where w_1 is the vertex which was the internal vertex of K_{1,n_2} and it lies in $(n_1 - 1)^{th}$ copy of K_{1,n_2} in $G \Box H$ and w_2 is the vertex which was the pendant vertex of K_{1,n_2} not in S and lies in n_1^{th} copy of K_{1,n_2} in $G \Box H$ be the set such that $V - S'$ is disconnected. Thus S' is the split edge geodetic set of $G\Box H$. Therefore

$$\begin{aligned} g_{1s}(G\Box H) &= |S'| \\ &= |S| + \{w_1\} + \{w_2\} \\ &= n_2 + 2. \end{aligned}$$

Theorem 5.2 : Let $C_{n_1}, n_1 \geq 4$ be any cycle and P_{n_2} be any path and $n_1 \geq n_2$. Then

$$g_{1s}(C_{n_1}\Box P_{n_2}) = \begin{cases} 4 & \text{if } n_1 \geq 3 \text{ is even, } n_2 \geq 3 \\ 5 & \text{if } n_1 = n_2 \text{ and is odd} \\ 6 & \text{if } n_1 \geq 7, 9, 11, \cdots, 3n_1 + 4 \end{cases}.$$

Proof : Consider $G = C_{n_1}\Box P_{n_2}$. Let G be the graph obtained by n_1 copies of P_{n_2}. Let $V = \{v_1, v_2, \cdots, v_{n_1}\}$ and $W = \{w_1, w_2, \cdots, w_{n_2}\}$ be the vertex set of C_{n_1} and P_{n_2} respectively. Then $U = V \cup W$. We discuss the following cases:

Case 1 : Let $S = \{(v_1, w_1), (v_i, w_j)\}$ where $d\{(v_1, w_1), (v_i, w_j)\} = diam(G)$ be the edge geodetic set of G. But $V - S$ is connected. Consider $S' = S \cup \{(v_2, w_2), (v_3, w_1)\}$. Clearly $V - S'$ is disconnected so that S' is the minimum split edge geodetic set of G. Hence $g_{1s}(G) = |S'| = 4$.

Case 2 : Let $S = \{(v_1, w_1), (v_i, w_j), (v_{n_1} w_{n_2})\}$ be the edge geodetic set of G such that $d\{(v_1, w_1), (v_i, w_j)\} = d\{(v_i, w_j), (v_{n_1}, w_1)\} = diam(G)$. Thus $I(S) = U$ and $V - S$ is connected. Consider $S' = S \cup \{(v_2, w_2), (v_3, w_1)\}$. Clearly $V - S'$ is an induced subgraph which has two components. Hence S' is the minimum split edge geodetic set of G. Therefore $g_{1s}(G) = |S'| = 5$.

Case 3 : Consider $S = \{(v_1, w_1), (v_i, w_j), (v_{n_1}, w_{n_2}), (v_{n_1}, w_1)\}$ be the edge geodetic set of G such that $d\{(v_1, w_1), (v_i, w_j)\} = d\{(v_i, w_j), (v_{n_1}, w_1)\} = diam(G)$. Thus $V - S$ is connected. Consider $S' = S \cup \{(v_2, w_2), (v_3, w_1)\}$. Clearly $V - S'$ is disconnected and hence S' is the minimum split edge geodetic set of G. Therefore $g_{1s}(G) = |S'| = 6$.

Theorem 5.3 : For integers $m \geq n \geq 2$, $g_{1s}(K_m\Box K_n) = g_1(K_m\Box K_n)$.

Proof : The proof follows from Theorem 3.5.

Observation 5.4 : Let T be the tree of order $n \geq 2$ with p number of end vertices. Consider H_x as the copy of K_n in $T\Box K_n$ with respect to an end vertex x of T. Then every geodesic contains an edge e where $e \in E(H_x)$.

Theorem 5.5 : Let p be the end vertices of non-trivial tree T and $n \geq 2$. Then $g_{1s}[T\Box K_n] = pn - p + 1$ for $p \leq n$.

Proof : Let $G = T\Box K_n$ be the cartesian product of non-trivial tree T and complete graph K_n. Consider $V(K_n) = \{a_1, a_2, \cdots, a_n\}$ and $V(T) = \{x_1, x_2, \cdots, x_p\}$.
Let S be the minimum edge geodetic set of G. Let the copy of K_n corresponding to the end vertex x of T be H_x. Now, let us claim that $|S \cap V(H_x)| \geq n - 1$.
In contrary, if $|S \cap V(H_x)| < n - 1$, then there should exist atleast two vertices $(x, y_1), (x, y_2)$ that are not in S. Since there is an isomorphism between H_x and K_n, it follows that $(x, y_1), (x, y_2)$ is an edge of H_x. Since S is an edge geodetic set of G, then by the Observtaion 5.4 $(x, y_1) \in S$ or $(x, y_2) \in S$, a contradiction. Thus $|S \cap V(H_x)| \geq n-1$. Since T has p end vertices, therefore $g_{1s}[T\Box K_n] = pn - p$. But $V - S$ is connected. Let $S' = S \cup (x_2, a_4)$ where (x_2, a_4) is a vertex in the second copy of G which is adjacent to (x_1, a_4) of first copy. Then $V - S$ is disconnected. Hence $g_{1s}(G) = |S'| = pn - p + 1$.

6. Corona

Theorem 6.1 : The corona for split edge geodetic number of path $P_{n_1}, n_1 \geq 3$ and path $P_{n_2}, n_2 \geq 2$ is $g_{1s}(P_{n_1} \circ P_{n_2}) = n_1 n_2 + 1$.

Proof : Let $G = P_{n_1} \circ P_{n_2}$ be the corona of paths P_{n_1} and P_{n_2} respectively. Let $B_1 = \{b_1, b_2, b_3, \cdots, b_{n_1}\}$ be block set such that $V_1(b_i(G)) = \{(u_1, w_i), (u_2, w_i), (u_3, w_i), \cdots, (u_{n_2}, w_i), w_i\}$ where $\{u_1, u_2, \cdots, u_{n_2}\} \in V(P_{n_2})$ and $w_i \in V(P_{n_1})$ for $1 \leq i \leq n_1$ are the vertex sets of P_{n_2} and P_{n_1} respectively. Let $B_2 = \{b'_1, b'_2, b'_3, \cdots, b_{n-1}$ be the block set such that $V[b'_j] = \{(w_i, w_{i+1})\}$. Let $S = V[b_i]$ be the edge geodetic set. But $V - S$ is connected, so S is not split edge geodetic set. Now, consider $S' = S \cup \{w_k\}, 1 < k < n_1$. Clearly $V - S'$ is an induced subgraph which has two components. Hence S' is split edge geodetic set. Thus

$$\begin{aligned} g_{1s}(G) &= |S'| \\ &= |S| + |\{w_k\}| \\ &= n_1 n_2 + 1. \end{aligned}$$

Theorem 6.2 : Let $G = F_{n_1}, n_1 \geq 2$ be a friendship graph and $H = P_{n_2}$ be a path of order $n_2, n_2 > 3$. Then $g_{1s}(G \circ H) = n_2(n_1 - 1) + 1$.

Proof : Consider $V(G) = \{v_1, v_2, \cdots, v_{2n_1+1}\}$ and $E(G) = \{e_1, e_2, e_3, \cdots, e_{3n_1}\}$ be the vertex and edge set of friendship graph F_n. Let $V(H) = \{u_1, u_2, \cdots, u_{n_2}\}$ be the vertex set of path P_{n_2}. Then $V(G\circ H) = V(G)\cup V(H)$. Let $S = \{w_1, w_2, \cdots, w_{n_2}\}$ where $\{w_1\}$ corresponds to vertex set of F_{n_1} of first copy of P_{n_2}, $\{w_2\}$ corresponds to vertex set of F_{n_2} of second copy of $P_{n_2}, \cdots, w_{n_2}$ corresponds to vertex set of F_{2n_1+1} of $(2n_1 + 1)^{th}$ copy of P_{n_2} excluding the common vertex to which n_1 copies of cycle graph C_3 is joined. Clearly S is an edge geodetic set. But $V - S$ is connected. Now choose vertex $\{u_k\}$, the internal vertex of one copy of P_{n_2} and $S' = S \cup \{v_k\}$ forms minimum split edge geodetic set of $G \circ H$ in which $V(G \circ H) - S'$ is disconnected. Hence

$$\begin{aligned} g_{1s}(G \circ H) &= |S'| \\ &= |S| + |\{u_k\}| \\ &= n_2(n_1 - 1) + 1. \end{aligned}$$

Corollary 6.3 : Let $G = P_{n_1} \circ C_{n_2}$ be the corona of path P_{n_1} and $C_{n_2}, n_2 > 2$. Then $g_{1s}(G) = n_1n_2 + 1$.

Corollary 6.4 : Let $G = C_{n_1} \circ C_{n_2}$ be the corona of cycle C_{n_1} and C_{n_2}, $n_1 \leq n_2$. Then $g_{1s}(G) = n_1n_2 + 2$.

7. Conclusion

In this paper we determine the split edge geodetic number for some graphs, cartesian product, corona and also we obtain the relation between edge geodetic and split edge geodetic number.

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