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SOME FIXED POINT THEOREMS ON COMPACT AND COMPLETE METRIC SPACES USING THE HYBRID GENERALIZED φ-WEAK CONTRACTION FOR TWO PAIR OF SELF MAPPINGS USING SEMI-COMPATIBILITY

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Abstract

In this paper we obtain some common fixed point theorems on compact and complete metric spaces for two pairs of self mappings, using the hybrid generalized φ-weak contraction. Our results improve, extend or generalize those results of Shambhu, Chhatrajit, Krishnanando [20] etc.

1. Introduction

The concept of contraction and weak contraction in metric fixed point Theory are found in a number of research papers viz [1], [2], [3], [4] [7], [8], [12], [20] etc. In 1997 Alber

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and Guerre - Delabriere [13] introduced weak contraction for single valued mappings in Hilbert Space and established existence of fixed points using φ-weak contraction on a complete metric space. Shambhu etal [20] proved existence of fixed point for a pair of self mappings on a compact metric space.

The aim of this paper is to prove existence of a unique common fixed points of a hybrid generalized φ-weak contraction for two pairs of self-mappings in compact and in complete metric spaces.

Before proving the main result we need the following definitions for our main results.

Definition 1.1 : Let (X, d) be a metric space. Let $f : X \to X$ be a mapping f is called.

(i) non-expansive of $d(fx, fy) \leq d(x, y), \ \forall \ \ x, y \in X$.

(ii) contractive if $d(fx, fy) < d(x, y), \quad \forall \ x, y \in X, x \neq y$,

(iii) a contraction if there exists a real number, $\lambda, 0 \leq \lambda < 1$ such that $d(fx, fy) \leq \alpha d(x, y), x \neq y$.

Definition 1.2 : A self-mapping $T : X \to X, (X, d)$ in a metric space is called a φ-weak contraction if for each $x, y \in X$ there exists a real valued function $\varphi[[0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$ for all $t \in [0, \infty)$ such that $\varphi(0) = 0$ and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)).$$

Definition 1.3 : Let (x, d) be a metric space and $T : X \to X$ be mapping. A point $x \in X$ is a fixed point of T if x is mapped onto itself under T i.e. if $Tx = x$.

Definition 1.4 : Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction defined on X. Then T has a unique fixed point in X.

Example 1.1 : The mapping $T : X \to R$ defined by $Tx = x^2$ has $x = 0$ and $x = 1$ as fixed points. This also shows that the fixed point, if exists may not be unique.

Definition 1.6 : Let $S, T : X \to X$ be two self-mappings. A point $x \in X$ is said to be a coincident point of S and T if $Sx = Tx$.

Example 1.2 : Let $Tx = x^2$, $Sx = x$ on the usual metric space (X, d). The points $x = 0$ and $x = 1$ are coincident points of S and T.

Example 1.3 : Consider the mappings $T, S : X \to R$ defined by $Tx = x^2$, $Sx = x^2, Sx = -x^2 \;\; \forall x \in (0,1] = X$ with the usual metric $d(x,y) = |x-y|$. Then T and S have no coincident point on X.

Example 1.4 : The mapping $T : X \to R, X = [0,1)$ defined by $Tx = \frac{1}{1+x} \;\; \forall\, x \in [0,1)$ is contractive but not a contraction. Moreover T has no fixed point. This examples shows that self-mappings may not have a fixed point.

2. Main Results

Theorem 2.1 : Let (X,d) be a compact metric space and let S, T, I and J be self mappings defined on X which satisfy the following for each $x, y \in X$:

(i) $S(X) \subseteq I(X)$ and $T(X) \subseteq J(X)$ (2.1.1)

(ii) $d(Sx,Ty) \leq C(x,y) - \varphi(C(x,y))$ (2.1.2)
where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous from the right with $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$ and

(iii) $C(x,y) = \max\left\{d(Ix,Jy), \frac{1}{2}[d(Ix,Sx) + d(Jy,Ty)], \frac{1}{2}[d(x,Ty) + d(Jy,Sx)]\right\}$ (2.1.3)
then if the pairs $\{S,J\}$ and $\{T,I\}$ are semi-compatible then they have coincident points which are the unique common fixed points of the mappings S, T, I and J.

Proof : Let x_0 be arbitrary. Define sequences $\{y_n\}$ and $\{x_n\}$ as

$$\begin{aligned} y_{2n} &= Sx_{2n} = Ix_{2n+1} \\ y_{2n+1} &= Tx_{2n+1} = Jx_{2n+2} \quad \text{for all} \;\; n = 0,1,2,3,.. \end{aligned}$$

such that

$$\begin{aligned} y_0 &= Sx_0 = Ix_1 \\ y_1 &= Tx_1 = Jx_2 \\ y_2 &= Sx_2 = Ix_3 \\ y_3 &= Tx_3 = Jx_4. \end{aligned}$$

Now,

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(Sx_{2n+1}, Tx_{2n}) \\ &\leq C(x_{2n+1}, x_{2n}) - \varphi(C(x_{2n+1}, x_{2n})) \\ &< C(x_{2n+1}, x_{2n}) \\ &= \max\Big\{d(Ix_{2n+1}, Jx_{2n}), \frac{1}{2}[d(Ix_{n+1}, Sx_{2n+1}) + d(Jx_{2n}, Tx_{2n})], \\ &\quad \frac{1}{2}[d(Ix_{2n+1}, Tx_{2n}) + d(Jx_{2n}, Sx_{2n+1})]\Big\} \\ &= \max\Big\{d(y_{2n}, y_{2n-1}), \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})], \\ &\quad \frac{1}{2}[d(y_{2n1}, y_{2n}) + d(y_{2n-1}, y_{2n+1})]\Big\} \end{aligned}$$

If $d(y_{2n+1}, y_{2n} > d(y_{2n}, y_{2n-1})$ then

$$C(x_{2n+1}, x_{2n}) = d(y_{2n+1}, y_{2n}) \tag{2.1.4}$$

Also,

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(Sx_{2n+1}, x_{2n}) \\ &\leq C(x_{2n+1}, x_{2n}) - \varphi(C(x_{2n+1}, x_{2n})) \\ &= d(y_{2n+1}, y_{2n} - \varphi(d(y_{2n+1}, y_{2n})) \\ &< d(y_{2n+1}, y_{2n}) \quad \text{as} \quad \varphi(t) > 0 \ \ \forall \ \ t > 0 \end{aligned} \tag{2.1.5}$$

which is a contradiction. Similarly we have another contradiction if n is taken an even number.

$$\therefore \quad d(_{2n+1}, y_{2n}) \leq d(y_{2n}, y_{2n-1}).$$

Now,$\{d(y_{2n+1}, y_{2n})\}$ and hence $\{d(y_{n+1}, y_n)\}_{n\geq 0}$ is a non-increasing sequence of positive real (R^+) which is bounded from below i.e. there exists a positive number r such that,

$$\lim_{n\to\infty} d(y_{2n+1}, y_{2n}) = \lim C(x_{2n+1}, x_{2n}) = r. \tag{2.1.6}$$

From (2.1.5), we have

$$\begin{aligned}
\lim_{n\to\infty} d(y_{2n+1}, y_{2n}) &\leq \lim_{n\to\infty} C(x_{2n+1}, x_{2n}) - \lim_{n\to\infty} \inf \varphi(C(x_{2n+1}, x_{2n})) \\
&\Rightarrow r \leq r - \varphi(r) \\
&\Rightarrow \varphi(r) \leq 0 \\
&\Rightarrow \varphi(r) = 0 \\
&\Rightarrow r = 0.
\end{aligned}$$

$$\therefore \lim_{n\to\infty} d(y_{2n+1}, y_{2n}) = 0. \tag{2.1.7}$$

To prove $\{y_{2n}\}$ or $\{y_n\}_{n\geq 0}$ is a Cauchy sequence. If not, there exists positive integers m and $n \geq N(\epsilon)$ for each positive number $\epsilon > 0$ such that

$$d(y_{n+1}, y_{m+1}) > \epsilon.$$

i.e.

$$\begin{aligned}
\epsilon < d(y_{n+1}, y_{m+1}) &= d(Sx_{n+1}, Tx_{m+1}) \\
&\leq C(x_{n+1}, x_{m+1}) - \varphi(C(x_{n+1}, x_{m+1})).
\end{aligned}$$

Letting $m, n \to \infty$ and using (2.1.6) we have, $\epsilon < 0$ which is a contradiction.

Thus, $\{y_n\}_{n\geq 0}$ is a Cauchy sequence.

Since X is compact and hence sequentially compact $\{y_n\}$ converges in X i.e. there exists. some $z \in X$ such that$y_{2n} \to z$ as $n \to \infty$.

i.e. $\lim_{n\to\infty} Sx_{2n}, \lim_{n\to\infty} Ix_{2n-1} \to z$.

If S, J are semi-compatible $\lim_{n\to\infty} JSx_{2n} = Sz \Rightarrow Jz = Sz$ i.e. z is a coincident point of J and S.

Similarly, T and I are semi-compatible implies $\lim_{n\to\infty} ITx_{2n} = Tz \Rightarrow Iz = Tz$

Let$Iz = Sz = Jz = Tx = t$ for some $t \in X$ then $t = z$. Assume that $t \neq z \Rightarrow d(t, z) \neq 0$.

Now,

$$
\begin{aligned}
d(t,z) &= d(\lim_{n\to\infty} y_{2n}, Tz) \\
&= \lim_{n\to\infty} (y_{2n}, Tz) \\
&= \lim_{n\to\infty} d(Sx_{2n}, Tz) \\
&< \lim_{n\to\infty} C(x_n, z) \\
&= \lim_{n\to\infty} \max\left\{ d(Ix_n, Jz), \frac{1}{2}[d(Ix_n, Sx_n) + d(Jz, Tz)], \frac{1}{2}[d(z,t) + d(t,z)] \right\} \\
&< \max\left\{ d(z,t), \frac{1}{2}[d(z,z) + d(t,t)], \frac{1}{2}[d(z,t) + d(t,z)] \right\} \\
&\to d(z,t) < d(z,t) \\
&\Rightarrow \text{ a contradiction.}
\end{aligned}
$$

Thus $z = t$. Hence $Sz = Iz = Jz = Tz = z'$. i.e. z is a common fixed point.
For Uniqueness, let, if possible there be another fixed point z' such that $Sz' = Iz' = Jz' = Tz = z'$ then $z = z$. If not $z \neq z'$ and hence $d(z,z') \neq 0$.
Now,

$$
\begin{aligned}
d(z',z') &= d(Sz, Tz') \\
&\leq C(z,z') - \varphi(C(z,z')) \\
&< C(z,z') \\
&= \max\left\{ d(Iz, Jz'), \frac{1}{2}[d(Iz, Sz') + d(Jz, Tz)], \frac{1}{2}[d(Iz, Tz') + d(Jz, Sz)] \right\} \\
&< \max\left\{ d(z,z'), \frac{1}{2}[d(z,z') + d(z,z')], \frac{1}{2}[d(z,z) + d(z',z)] \right\} \\
&= \max\left\{ d(z,z'), \frac{1}{2}s(z,z'), d(z,z') \right\}
\end{aligned}
$$

i.e. $d(z,z') < d(z,z')$ which is a contradiction.

Theorem 2.2 : Theorem 2.1, 2.2, 2.3, 2.4 etc. of [20] are particular cases of Theorem 2.1 above, on setting $I = J = I$, the Identity mapping on X.

Theorem 2.3 :Let (X,d) be a complete metric space and, let S, T, I and J are self mappings defined on Xsuch that for all $x, y \in X$ they satisfy.

(i) $S(X) \subseteq I(X)$ and $T(X) \subseteq J(X)$

(ii) $d(Sx, Ty) \leq C(x, y) - \varphi(C(x, y))$ wheere $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous from the right with $\varphi(t) > 0 \ \forall \ \ t \in [0, \infty)$ and $\varphi(0) = 0$.

(iii) $C(x, y) = \max \left\{ d(Ix, Jy), \frac{1}{2}[d(Ix, Sx) = d(Jy, Ty)], \frac{1}{2}[d(Ix, Ty) + d(Jy, Sx)] \right\}$, then if the pairs $\{S, J\}$ and T, I are semi-compatible then they have coincident points which are the unique common fixed points of S, I, I and J.

Theorem 2.4 : Let (X, d) be a sequentially compact metric space and let S, T, I and J be self mappings defined on X such that all the conditions (i), (ii) and (iii) held and one of the range space $I(X), J(X), S(X)$ as $T(X)$ is a closed subspace X them if the pairs $\{S, I\}$ and $\{T, J\}$ are semi-compatible then they have coincident points in X which are the unique common fixed points of S, T, I and J.

Proof : The details of the proofs of Theorem 2.2, 2.3, 2.4 etc. are omitted.

Theorem 2.5 : $X = [0, 1] \subseteq R$

$$\begin{aligned} Sx &= Jx = \begin{cases} x, & x \in [0, \frac{1}{3}) \\ \\ \frac{1}{3}, & x \in [\frac{1}{3}, 1] \end{cases} \\ Tx &= Ix = \begin{cases} 1 - 2x, & x \in [0, \frac{1}{3}) \\ \\ \frac{1}{3}, & x \in [\frac{1}{3}, 1] \end{cases} \end{aligned}$$

Choose $\{x_n\}, x_n \in X$ s.t $x_n \to \frac{1}{3}$.

$$\begin{aligned} \lim_{n\to\infty} Ix_n &= \lim_{n\to\infty} Sx_n = \frac{1}{3}. \\ \lim_{n\to\infty} Tx_n &= \lim_{n\to\infty} Jx_n = \frac{1}{3}. \\ \lim_{n\to\infty} JSx_n &= J\frac{1}{3} = \frac{1}{3} = S\frac{1}{3} \\ \lim_{n\to\infty} ITx_n &= 1\frac{1}{3} = \frac{1}{3} = T\frac{1}{3}. \end{aligned}$$

Thus, the pairs $\{S, J\}$ and $\{T, I\}$ are semi-compatible and continuous. at $x = \frac{1}{3}$. Further $x = \frac{1}{3}$ is the unique common fixed point of the mappings S, I, T and J.

Example 2.6 : Define mapping , S, T, I and J on $X = [0, \infty)$ by

$$Sx = Ix = \begin{cases} 2 + 3x, & x \in [0, 1) \\ \\ 2, & x \in [1, \infty) \end{cases}$$

$$Tx = Jx = \begin{cases} 2, & x \in [0, 1) \\ 2 + 3x, & x \in [1, \infty) \end{cases}$$

Choose $\{x_n\}$ such that $x_n \to \frac{1}{n} \to 0$ as $n \to \infty$.

Thus $\lim_{n\to\infty} Sx_n = 2 = \lim_{n\to\infty} Jx_n$.

Also, $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = 2$.

Now, $\lim_{n\to\infty} JSx_n = J2 = 26 \neq S2$.

Therefore, $\{S, J\}$ is not semi-compatible.

Again $\lim_{n\to\infty} ITx_n = 2 = 2 \neq T2 = 26$.

Hence $\{T, I\}$ is also not semi-compatible.

The mappings S, I, T and J don't have a common fixed point in X. S, I, T and J are discontinuous at $x = 1$.

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