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AN INTEGRAL INVOLVING THE PRODUCT OF AN INCOMPLETE GAMMA FUNCTION, GENERALIZED STRUVE'S FUNCTION AND I-FUNCTION OF r-VARIABLES

M. SUNITHA[1], P. C. SREENIVAS[2] AND T. M. VASUDEVAN NAMBISAN[3]

[1] Department of Mathematics,
Govt. Brennen College, Dharmadam, Thalassery,
Kannur Dist, Kerala -670106, India
[2] Department of Mathematics, Payyanur College, Payyanur,
Kannur Dist, Kerala- 670327, India
[3] Retired Professor and Head, Department of Mathematics,
NAS College, Kanhangad, Kerala, India

Abstract

The object of this paper is to evaluate an integral involving the product of an Incomplete Gamma function, Generalized Struve's function and the I-function of several complex variables. On specializing the parameters similar results can be derived in the case of I-functions of two variables and H functions of r-variables, which include the result proved by Shahul Hameed [5, p. 70].

1. Introduction

Notations used:

${}_1(a_j; \alpha_j, A_j)_p$ stands for $(a_1; \alpha_1, A_1), (a_2; \alpha_2, A_2), \cdots, (a_p; \alpha_p, A_p)$.

Key Words : *Incomplete Gamma function, I-function of two and several complex variables, Multivariable H functions, Generalized Struve's function.*

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The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, Nambisan and Santha Kumari [4] is defined and represented in the following manner:

$$I[z_1, \cdots, z_r]$$

$$I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots;p_r,q_r}\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P :{}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \cdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q :\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)\ z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r, \tag{1.1}$$

where $\phi(s_1, \cdots, s_r)$ and $\theta_i(s_i), i = 1, 2, \cdots, r$ are given by,

$$\phi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{N}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod_{j=1}^{Q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i\right)\prod_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)}, \tag{1.2}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}.$$

Also $z_i \neq -\ (i = 1, \cdots, r)$, $\omega = \sqrt{-1}, m_j, n_j, p_j, q_j\ (j = 1, \cdots, r)$, N, P, Q are non-negative integers such that $0 \leq N \leq P$, $Q \geq 0$, $0 \leq m_j \leq q_j$, $0 \leq n_j \leq p_j$ $(j = 1, 2, \cdots, r)$ (not all zero simultaneously). $\alpha_j^{(i)}\ (j = 1, 2, \cdots, P, i = 1, 2, \cdots, r)$, $\beta_j^{(i)}(j = 1, 2, \cdots, Q, i = 1, 2, \cdots, r)$, $\gamma_j^{(i)}\ (j = 1, 2, \cdots, p_i, i = 1, 2, \cdots, r)$ and $\delta_j^{(i)}\ (j = 1, 2, \cdots, q_j, i = 1, 2, \cdots, r)$ are positive numbers. $a_j\ (j = 1, 2, \cdots, P)$, $b_j\ (i = 1, 2, \cdots, Q)$, $c_j^{(i)}\ (j = 1, 2, \cdots, p_i, i = 1, 2, \cdots, r)$ and $d_j^{(i)}\ (j = 1, 2, \cdots, q_i, i = 1, 2, \cdots, r)$ are complex numbers. The exponents $A_j\ (j = 1, 2, \cdots, P)$, $B_j\ (j = 1, 2, \cdots, Q)$, $C_j^{(i)}\ (j = 1, 2, \cdots, p_i, i = 1, 2, \cdots, r)$ and $D_j^{(i)}\ (j = 1, 2, \cdots, q_i, i = 1, 2, \cdots, r)$ of various gamma functions may take non integer values. The I-function of r variables is analytic if

$$\Psi_i = \sum_{j=1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)}\delta_j^{(i)} \leq 0, i = 1, 2, \cdots, r.$$

The integral (1.1) converges absolutely if $|arg(z_i)| < \frac{1}{2} < \pi\Delta_i, i = 1, 2, \cdots, r$ where

$$\begin{aligned}\Delta_i \ &=\ -\sum_{j=n+1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)}\delta_j^{(i)} \\ &\quad + \sum_{j=1}^{n_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\gamma_j^{(i)} > 0.\end{aligned}$$

The I -function of r-variables is also defined and represented in the following manner:

$$\overline{I}[z_1, \cdots, z_r]$$

$$I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \cdots \\ z_r \end{array}\left|\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)}A_j)_P :_1 (c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1}; \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q :\ {}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1,m+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1}; \\ \cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};1)_{m_r,m_r+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_r} \end{array}\right.\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\overline{\theta}_1(s_1)\cdots\overline{\theta}_r(s_r)\phi(s_1,\cdots,s_r)\ z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r, \tag{1.3}$$

where

$$\overline{\theta}_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}.$$

The integral converges absolutely if $|arg(z_i)| < \frac{1}{2}\pi\ \Delta_i', i = 1, 2, \cdots, r$, where

$$\Delta_i' = -\sum_{j=n+1}^{P}A_j\alpha_j^{(i)} - \sum_{j=1}^{Q}B_j\beta_j^{(i)} + \sum_{j=1}^{m_i}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i}D_j^{(i)}\delta_j^{(i)} + \sum_{j=1}^{n_i}C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}C_j^{(i)}\gamma_j^{(i)} > 0. \tag{1.4}$$

The Generalized Struve's function defined by Kanth [2, p. 18]

$$H^{\lambda,k}_{v,y,\mu}(z) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{r}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)} \tag{1.5}$$

where $Re(k) > 0,\ Re(\lambda) > 0,\ Re(y) > 0,\ Re(v+\mu) > 0.$

The Incomplete Gamma function is defined by $\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}dt$ Gupta and Agarwal [1]. If

$$h_1(\lambda) = L(e^{bx}h_2(e^{ax});\lambda) \tag{1.6}$$

$$h_2(\lambda) = L(f(x);\lambda) \tag{1.7}$$

Then

$$h_1(\lambda) = \frac{1}{a}\int_0^\infty x^{\frac{\lambda-b}{a}}\Gamma\left(\frac{b-\lambda}{a}, x\right)f(x)dx. \tag{1.8}$$

Sreenivas, Mambisan and Suitha [7]

$$\int_0^\infty e^{-\eta x}\overline{I}[z_1x^{\sigma_1}, z_2x^{\sigma_2}, \cdots, z_rx^{\sigma_r}]H^{\lambda,k}_{v,y,\mu}(z\ x^\rho)dx = \frac{1}{\eta}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2\eta^\rho}\right)^{v+2m+1}}{\times}\overline{I}^{0,N+1:m_1,n_1;\cdots;m_r,n_r}_{P+1,Q:p_1,q_1;\cdots;p_r,q_r}$$

$$\left[\begin{array}{c} \frac{z_1}{\eta^{\sigma_1}} \\ \cdots \\ \frac{z_r}{\eta^{\sigma_r}} \end{array}\left|\begin{array}{l} (-\rho(v+2m_1);\sigma_1,\sigma_r;1)(a_j;\alpha_j^{(i)},\cdot,\alpha_j^{(r)};A_j)_P;\ {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q;\ {}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1,m_i+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_i};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};1)_{m_r,m_r+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_r} \end{array}\right.\right] \tag{1.9}$$

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$$\int_0^\infty x^{\frac{\eta-b}{a}}\Gamma\left(\frac{b-\eta}{a},x\right)\overline{I}[z_1x^{\sigma_1},z_2x^{\sigma_2},\cdots,z_rx^{\sigma_r}]H^{\lambda,k}_{v,y,\mu}(zx^\rho)dx$$
$$= a\sum_{m=0}^{\infty}G(m\times\overline{I}^{0,N+2m_1,n_1;\cdots;m_r,n_r}_{P+2,Q+1:p_1,q_1;\cdots;p_r,q_r}$$
$$\left[\begin{array}{c|l}\frac{z_1}{e^{ax\sigma_1}} & (-\rho(v+2m+1);\sigma_2,\cdots\sigma_r;1),(1-\eta+b-a;a\sigma_1,\cdots a\sigma_r;1),\\ \vdots & \\ \frac{z_r}{\sigma^{ax\sigma_r}} & (-\eta+b-a;a\sigma_1,\cdots,a\sigma_r;1),\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q\end{array}\right.$$
$$\left.\begin{array}{l}{}_1(aj;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_p:\ {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;(c_j^{(r)}.\gamma_j^{(r)};C_j^{(r)})_{p_r}\\ {}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1,m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};1)_{m_r,m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right] \tag{2.1}$$

where

$$G(m)=\frac{-1)^m\left(\frac{z}{2e^{ax\rho}}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)} \tag{2.2}$$

provided $Re(\eta)>0, Re(k)>0, Re(\lambda)>0, Re(y)>0, Re(v+m\mu)>0, \sigma_i>0, i=1,2,\cdots,r$

$$Re\left(1+\sum_{i=1}^{r}\frac{\sigma_i d_j^{(i)}}{\delta_j^{(i)}}\right)>0, j=1,2,\cdots,m, |arg(z_i)|<\frac{1}{2}\pi\Delta'_o, i=1,2,\cdots,r,$$

where Δ is given by (1.4).

Proof : Let

$$f(x)=\overline{I}[z_1,x^{\sigma_1},z_2x^{\sigma_2},\cdots,x^{\sigma_r}]H^{\lambda,k}_{v,y,\mu}(z\ x^\rho).$$

On using (1.7) and (1.9)

$$h_2(\eta)=\int_0^\infty e^{-\eta x}f(x)dx=int_0^\infty e^{-\eta x}[z_1x^{\sigma_1},z_2x^{\sigma_2},\cdots,z_r,x^{\sigma_r}]H^{\lambda,k}_{v,y,\mu}(z\ x^\rho)dx$$
$$=\frac{1}{\eta}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2\eta^\rho}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)}\times\overline{I}^{0,N+2m_1,n_1;\cdots;m_r,n_r}_{P+2,Q+1:p_1,q_1;\cdots;p_r,q_r}$$
$$\left[\begin{array}{c|l}\frac{z_1}{\eta^{\sigma_1}} & (-\rho(v+2m+1);\sigma_1,\cdots\sigma_r;1),(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}\\ \vdots & \\ \frac{z_r}{\eta^{\sigma_r}} & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1,m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};1)_{m_r,m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right]$$

(2.3)

Again using (2.3) in (1.6) and (1.8) to get

$$h_1(\eta) = \int_0^\infty e^{-\eta x} e^{bx} h_2(e^{ax}) dx$$

$$= \int_0^\infty e^{-\eta x} e^{bx} \frac{1}{e^{ax}} H^{\lambda,k}_{v,y,\mu}\left(\frac{z}{e^{ax\rho}}\right) \times \bar{I}^{0,N+2m_1,n_1;\cdots;m_r,n_r}_{P+2,Q+1:p_1,q_1;\cdots;p_r,q_r}$$

$$\left[\begin{array}{c|c} \frac{z_1}{\eta^{ax\sigma_1}} & (-\rho(v+2m+1);\sigma_1,\cdots\sigma_r;1),(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ \frac{z_r}{\eta^{ax\sigma_r}} & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1,m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};1)_{m_r,m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right] dx$$

$$= \int_0^\infty e^{-\eta x} e^{bx} \frac{1}{e^{ax}} \left\{\left(\frac{1}{2\pi\omega}\right)^r \int_{L_1}\cdots\int_{L_r} \phi(s_1,\cdots s_r)\bar{\theta}_1(s_1)\cdots\bar{\theta}(s_r)\right.$$

$$\left.\times\Gamma\left(1+\rho(v+2m+1)+\sum_{i=1}^r \sigma_i s_i\right) \times \left(\frac{z_1}{e^{a\sigma_1 x}}\right)^{S_1}\cdots\left(\frac{z_r}{e^{a\sigma_r x}}\right) ds_1\cdots ds_r\right\} dx$$

$$= \sum_{m=0}^\infty \frac{(-1)^m\left(\frac{z}{2e^{a\rho x}}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)} \left(\frac{1}{2\pi\omega}\right)^r \int_{L_1}\cdots\int_{L_r} \phi(s_1,\cdots,s_r)\bar{\theta}_1(s_1)\cdots\overline{\theta_r}(s_r) z_1^{S_1}\cdots z_r^{S_r} \times$$

$$\left[\int_0^\infty e^{-\eta x} e^{(b-a)x} e^{-a\left(\sum_{i=1}^r \sigma_i s_i\right)x} dx\right] \times \Gamma\left(1+\rho(v+2m+1)+\sum_{i=1}^r \sigma_i s_i\right) ds_1\cdots ds_r$$

(by changing the order of integration and summation which are justified under the given conditions)

$$= \sum_{m=0}^\infty \frac{(-1)^m\left(\frac{z}{2e^{a\rho x}}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)} \left(\frac{1}{2\pi\omega}\right)^r \int_{L_1}\cdots\int_{L_r} \phi(s_1,\cdots,s_r)\bar{\theta}_1(s_1)\cdots\bar{\theta}_r(s_r) z_1^{S_1}\cdots z_r^{S_r} \times$$

$$\frac{1}{n-b+a+a\sum_{i=1}^r \sigma_i s_i} \times \Gamma\left(1+\rho(v+2m+1)+\sum_{i=1}^r \sigma_i s_i\right) ds_1\cdots ds_r$$

$$= \sum_{m=0}^\infty \frac{(-1)^m\left(\frac{z}{2e^{a\rho x}}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)} \left(\frac{1}{2\pi\omega}\right)^r \int_{L_1}\cdots\int_{L_r} \phi(s_1,\cdots,s_r)\bar{\theta})1(s_1)\cdots\bar{\theta}(s_r) z_1^{S_1}\cdots z_r^{S_r} \times$$

$$\frac{\Gamma\left(\eta-b-a+a\sum_{i=1}^r \sigma_i s_i\right)}{\Gamma\left(1-\eta-b+a+a\sum_{i=1}^r \sigma_i s_i\right)} \times \Gamma\left(1+\rho(v+2m+1)+\sum_{i=1}^r \sigma_i s_i\right) ds_1\cdots ds_r$$

from which the result is obtained using (1.3), (1.8) and (2.2).

Special Cases

1. In (2.1), put $a = 1$ and replace $b + \eta$ to get

$$\int_0^\infty x^{-b}\Gamma(b,x)\bar{I}[z_1 x^{\sigma_1}, z_2 x^{\sigma_2},\cdots,z_r x^{\sigma_r}] H^{\lambda k}_{v,y,\mu)}(z\ x^\rho) dx$$

$$= \sum_{m=0}^\infty \Gamma(m)\Gamma(v+\lambda m+\mu) \times \bar{I}^{0,N+2m_1,n_1;\cdots;m_r,n_r}_{P+2,Q+1:p_1,q_1;\cdots;p_r,q_r} \left[\begin{array}{c|c} \frac{z_1}{\eta^{ax\sigma_1}} & (-\rho(v+2m+1);\sigma_1,\cdots\sigma_r;1),(b;\sigma_1,\cdots,\sigma_r;1), \\ \vdots & \\ \frac{z_r}{\sigma\eta^{\sigma_r}} & (b-1);\sigma_1,\cdots,\sigma_r;1),\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q \end{array}\right.$$

$$\left.\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)};1)_{m_1,m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};1)_{m_r,m_r+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_r} \end{array}\right]$$

provided the given conditions are satisfied.

2. Taking $r = 2$, the result reduces to the result involving $\overline{I}$ function of 2 variables

$$\int_0^\infty x^{\frac{n-b}{a}} \Gamma\left(\frac{b-\eta}{a}, x\right) \overline{I}[z_1 x^{\sigma_1}, z_2 x^{\sigma_2}], H^{\lambda,k}_{v,y,\mu}(z\ x^\rho)dx$$
$$= a \sum_{m=0}^\infty \Gamma(m) \times \overline{I}^{0,N+2m_1,n_1;\cdots;m_r,n_r}_{P+2,Q+1:p_1,q_1;\cdots;p_r,q_r} \left[\begin{array}{c} \frac{z_1}{e^{ax\sigma_1}} \\ \vdots \\ \frac{z_r}{e^{ax\sigma^2}} \end{array} \middle| \begin{array}{l} (-\rho(v+2m+1);\sigma_1,\sigma_2;1),(1-\eta+b-a' a\sigma_1, a\sigma_2;1), \\ (-\eta+b-1;a\sigma_1,a\sigma_2;1),\ {}_1(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j)_Q \end{array} \right.$$
$$\left. \begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j)_P :\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};(c_j^{(2)},\gamma_j^{(2)},\gamma_j^{(2)};C_j^{(r)})_{p_2} \\ {}_1(d_j^{(1)};\delta_j^{(1)};1)_{m_1,m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};(d_j^{(2)},\delta_j^{(2)};1)_{m_2,m_2+1}(d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_2} \end{array} \right] \tag{2.4}$$

provided $Re(\eta) > 0, Re(k) > 0, Re(\lambda) > 0, Re(y) > 0, Re(v+\mu) > 0, \sigma_i > 0, i = 1, 2.$

$$Re\left(1 + \sum_{i=1}^{2} \frac{\sigma_i d_j^{(i)}}{\delta_j^{(i)}}\right) > 0, \quad j = 1, 2, \cdots, m_2,$$

$|arg(z_i)| < \frac{1}{2}\pi \quad \Delta_i', i = 1, 2$, where Δ_i is given by (1.4).

3. When all the exponents are equal to unity, (2.1) reduces to the formula involving H function of r- variables

$$\int_0^\infty x^{\frac{n-b}{a}} \Gamma\left(\frac{b-\eta}{a}, x\right) H[z_1 x^{\sigma_1}, z_2 x^{\sigma_2}, \cdots, z_r x^{\sigma_r}] H^{\lambda,k}_{v,y,\mu}(z\ x^\rho)dx$$
$$= a \sum_{m=0}^\infty \Gamma(m) \times H^{0,N+2m_1,n_1;\cdots;m_r,n_r}_{P+2,Q+1:p_1,q_1;\cdots;p_r,q_r} \left[\begin{array}{c} \frac{z_1}{e^{ax\sigma_1}} \\ \vdots \\ \frac{z_r}{e^{ax\sigma^2}} \end{array} \middle| \begin{array}{l} (-\rho(v+2m+1);\sigma_1,\cdots,\sigma_r;1),(1-\eta+b-a;a\sigma_1,\cdots,a\sigma_r;1), \\ (-\eta+b-a;a\sigma_1,\cdots,a\sigma_r;1),\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q \end{array} \right.$$
$$\left. \begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j)_P :\ {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};1)_P;\ {}_1(c_j^{(1)},\gamma_j^{(1)};1)_{p_1};\cdots;\ {}_1(c_j^{(r)};\gamma_j^{(r)};1)_{p_r} \\ {}_1(d_j^{(1)};\delta_j^{(1)};1)_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};1)_{q_r} \end{array} \right] \tag{2.5}$$

provided $Re(\eta) > 0, Re(k) > 0, Re(\lambda) > 0, Re(y) > 0, Re(v+\mu) > 0, \sigma_i > 0, i = 1, 2, \cdots, r.$

$$Re\left(1 + \sum_{i=1}^{r} \frac{\sigma_i d_j^{(i)}}{\delta_j^{(i)}}\right) > 0, \quad j = 1, 2, \cdots, m_r,$$

$|arg(z_i)| < \frac{1}{2}\pi \quad \Delta_i', i = 1, 2$, where Δ_i is given by (1.4).

4. When $r = 2$ (2.4) reduces to corresponding result for H function of 2 variables given by Shahul Hammed [5, p. 70].

$$\int_0^\infty x^{\frac{n-b}{a}}\Gamma\left(\frac{b-\eta}{a},x\right)H[z_1x^{\sigma_1},z_2x^{\sigma_2}]H_{v,y,\mu}^{\lambda,k}(z\ x^{\rho})dx$$

$$= a\sum_{m=0}^{\infty}\Gamma(m)\times H_{P+2,Q+1:p_1,q_1;p_2,q_2}^{0,N+2m_1,n_1;m_2,n_2}\left[\begin{array}{c|l} \frac{z_1}{e^{ax\sigma_1}} & (-\rho(v+2m+1);\sigma_1,si_2;1),(1-\eta+b-a;a\sigma_1,a\sigma_2;1), \\ \frac{z_2}{e^{ax\sigma^2}} & (-\eta+b-a;a\sigma_1,a\sigma_2;1),\ _1(b_j;\beta_j^{(1)},\beta_j^{(2)};1)_Q \end{array}\right.$$
$$\left.\begin{array}{l} _1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};1)_P:\ _1(c_j;\gamma_j^{(1)};1)_{p_1};\ _1(c_j^{(2)},\gamma_j^{(2)};1)_{p_2}; \\ _1(d_j^{(1)};\delta_j^{(1)};1)_{q_1};\ _1(d_j^{(2)},\delta_j^{(2)};1)_{q_2} \end{array}\right] \tag{2.6}$$

provided $Re(\eta)>0, Re(k)>0, Re(\lambda)>0, Re(y)>0, Re(v+\mu)>0, \sigma_i>0, i=1,2.$

$$Re\left(1+\sum_{i=1}^{r}\frac{\sigma_i d_j^{(i)}}{\delta_j^{(i)}}\right)>0, \quad j=1,2,\cdots,m_2,$$

$|arg(z_i)|<\frac{1}{2}\pi\ \ \Delta_i', i=1,2,$ where Δ_i is given by (1.4).

References

[1] Gupta K.C. and Agrawal S. M.,Substitution theorems for integral transforms with symmetric Kernels', Indian Journal of pure and applied Maths., 19(6) (June 1988), 567-574.

[2] Kanth B. N., Jour. Inst. Sci., Nepal, 3(1) (1980), 99.

[3] Prasanna Kumari M. R., Applications of Integral Transforms, PhD Thesis, University of Calicut (1992).

[4] Prathima J., Vasudevan Nambisan T. M. and Shantha Kumari K.,A study of I-function of several Complex Variables, International Journal of Engineering Mathematics, Volume 2014, Article ID 931395, http://dx.doi.org/10.1155/2014/931395 (January 2014).

[5] Shahul Hameed K. P.,Investigations in Generalized Hypergeometric Functions, Ph.D. Thesis, Kannur University (2009).

[6] Shantha Kumari K., Investigations in I-functions of Two and Several Complex Variables, PhD Thesis, Sri Chandrashekarendra Saraswathi Viswa Maha Vidyalaya, Kanchipuram, (2014).

[7] Sreenivas P. C., Vasudevan Nambisan T. M. and Sunitha M., On the laplace transform of the functions involving the product of generalizes Struve's function and the I-function of r-variables, International Journal of Mathematical Sciences & Engineering Applications, 11(III) (December 2017), 179-184.

[8] Srivastava H. M., Gupta, K. C. and Goyal S. P., The H-functions of One and Two variables with Applications, South Asian Publishers, New Delhi, (1982).