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THINNING - MANIFESTATIONS OF GEOMETRIC SUMS IN STOCHASTIC MODELS

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Abstract

p-thinning of renewal processes is closely related to geometric sum of non-negative r.v s. Harris distribution is a generalization of the geometric distribution. Here we review how geometric sums and Harris sums manifest in AR(1) models. Harris thinning of random walks is introduced generalizing p-thinning of random walks. It is observed that Harris thinning of renewal process is not a renewal process. Results in these thinning mechanisms are arrived at in a unified manner.

1. Introduction

p-thinning of point processes arises in many practical situations such as counters, crime and accident data, reported cases of HIV, TB etc. where only some of the events are

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observed or reported. First paper on this is by Renyi (1956) who characterized Poisson process as the only process with finite mean that is invariant under rarefaction (p-thinning) and contraction (change of scale) applied together. p-thinning of point processes occur when each point is retained with a constant probability p and deleted with probability $(1-p)$, independent of all other points and independent of the point process itself.

In renewal point processes the inter-arrival times of the p-thinned process is the geometric sum of the inter-arrival times of the original process. The result by Renyi (1956) is the characterization of exponential distributions by the property of invariance of geometric sums up-to a scale change. Sandhya (1991 a, b) extended the result of Renyi (1956) by proving that a renewal process is invariant under p-thinning and contraction (i) for some $p \in (0,1)$ iff its inter-arrival time is semi-Mittag-Letter (semi-ML), (ii) for every $p \in (0,1)$ iff its inter-arrival time is ML and (iii) if mean exists then exponential, that is the process is Poisson. For more on p-thinning see Yannaros (1987, 1989), Sandhya (1991 a, b), Teke and Deshmukh (2008) and Deshmukh and Teke (2014). p-thinning of random walks was introduced and studied by Yannaros (1991) where the thinning mechanism results in a random walk with steps which is the geometric sum of the steps of the original random walk.

Pillai (1985) introduced and studied semi-α-Laplace (geometrically semi-stable) distributions with characteristic function (CF) of the form $\frac{1}{1+\psi(t)}$ where $p\psi(t) = \psi(ct), 0 < c < 1 < \frac{1}{p}$ and 'α' is the unique solution to $c^{\alpha} = p$, $\alpha \in (0,2]$. If $p\psi(t) = \psi(ct)$ is true for two values p_1 and p_2, such that $\log(p_1) = \log(p_2)$ is irrational, then semi-α-Laplace becomes α-Laplace (geometrically stable) distribution with CF $1/\{1+\beta|t|^{\alpha}\}, \beta > 0$. If we assume the existence of second moment ($\alpha = 2$), then the distribution is Laplace. Note that semi-ML, ML and exponential ($\alpha = 1$) distributions are positive-valued analogues of semi-α-Laplace, α-Laplace and Laplace distributions with similar structures for their Laplace transforms (LT), $\alpha \in (0,1]$.

A renewal process $\{N(t); t \geq 0\}$ is said to be a Cox process if there exists a process $\{N_p(t), t \geq 0\}$ such that $\{N(t)\}$ is the p-thinned process of $\{N_p(t)\}$ for every $p \in (0,1)$. Then $N_p(t)$ is the p-inverse of $N(t)$; $\forall p \in (0,1)$. Kingman (1964) (also, Sandhya (1991a)) showed that a renewal process is Cox iff its inter-arrival time is geometrically infinitely divisible (GID), though the former didn't use the term GID. By Klebanov et al. (1984), a r.v Y is GID if for every $p \in (0,1)$, it is a geometric sum

$$Y = X_1 + \cdots + X_{\nu p}$$

where $X_1, X_2, \cdots$ are i.i.d r.v s independent of ν_p for each $p \in (0,1)$, and ν_p has a geometric distribution with mean $1/p$. The CF of a GID distribution has the representation $\frac{1}{1+\varphi(t)}$ such that $e^{-\varphi(t)}$ is infinitely divisible (ID). Thus, for a Cox and renewal process, the LT of inter-arrival times is $\frac{1}{1+\varphi(\lambda)}$ where $\varphi(\lambda)$ has complete monotone derivative (Bernstein function) with $\varphi(0) = 0$.

Results on geometric sums relevant in this context are;

Theorem 1.1 (Kingman (1964), Sandhya (1991 a)) : A renewal process is Cox iff its inter-arrival time is GID.

Theorem 1.2 (Sandhya (1991 b)) : The distribution of a r.v is invariant (up to a scale change $c \in (0,1)$), under a geometric sum (i) for some $p \in (0,1)$ iff it is semi-α-Laplace (ii) for every $p \in (0,1)$ iff it is α-Laplace and (iii) if second moment exists ($\alpha = 2$) then Laplace.

For non-negative r.v.s the analogous conclusions are (Sandhya (1991 a)): (i) semi-ML (ii) ML and (iii) if first moment exists ($\alpha = 1$) then exponential.

Harris (m, k) distribution (Harris (1948)) was studied in some detail by Sherly (2008), Sandhya et al. (2008) and Lovely (2015) and the notion of Harris infinite divisibility (HID) was introduced in Satheesh et al. (2008).

Definition 1.3 : A r.v Y is HID if for every $p \in (0,1)$, it is a Harris sum

$$Y = X_1 + \cdots + X_{H_p} \tag{1}$$

where $X_1, X_2 \cdots$ are i.i.d r.v.s independent of H_p for each $p \in (0,1)$ and H_p has a Harris (m, k) distribution with mean $m = 1/p$ and probability generating function

$$P(s) = \left(\frac{s^k}{m - (m-1)s^k}\right)^{1/k}, \quad k \geq 1 \ \ integer.$$

For $k = 1$, Harris distribution is the geometric distribution with mean $1/p$. Putting $p = 1/m$ and taking CFs on both sides of (1) we get; A CF $h(t)$ is HID if for every $p \in (0,1)$ and for a fixed integer $k \geq 1$ there exists another CF $\phi_p(t)$ such that

$$h(t) = \left(\frac{p\phi_p^k(t)}{1 - (1-p)\phi_p^k(t)}\right)^{1/k}.$$

Consequently the CF of a HID distribution has the form, Satheesh et al. (2008),

$$h(t) = \left(\frac{1}{1+\varphi(t)}\right)^{1/k},$$

where $e^{-\varphi(t)}$ is ID or $\frac{1}{1+\varphi(t)}$ is GID. The LT of a positive valued HID r.v is $\left(\frac{1}{1+\varphi(\lambda)}\right)^{1/k}$, such that the LT $e^{-\varphi(\lambda)}$ is ID.

Notice that Harris sum is geometric sum of fixed sum of k r.v s and then looking at the distribution of the components in the fixed sum. Results on Harris sums relevant in this context are;

Theorem 1.4 (Satheesh et al. (2002)) : The distribution of a r.v is invariant (up-to a scale change $c \in (0,1)$), under a Harris sum (i) for some $p \in (0,1)$ iff it is generalized semi-α-Laplace with CF $\left\{\frac{1}{\psi(t)}\right\}^{1/k}$, $k > 0$ integer, where $p\psi(t) = \psi(ct), 0 < c < 1 < \frac{1}{p}$ and 'α' is the unique solution to $c^\alpha = p$, $\alpha \in (0,2]$, (ii) for every $p \in (0,1)$ iff it is generalized α-Laplace with CF $1/\{1+\beta|t|^\alpha\}^{1/k}$, $\beta > 0$ and (iii) if second moment exists ($\alpha = 2$) then generalized Laplace.

For non-negative r.vs the analogous conclusions are: (i) generalized semi-ML (ii) generalized ML and (iii) if first moment exists ($\alpha = 1$) then gamma (Harris (1948)).

Certain problems of interest here were invariance of geometric sums and Harris sums (i) for some $p\underline{(}0,1)$, (ii) for every $p \in (0,1)$ and (iii) whether existence of mean/ variance of distributions is assumed. While the relation between geometric sum and p-thinning of renewal process and random walk is more direct, its manifestation in AR(1) models is not so. It is also interesting to know the manifestations of Harris sums in these stochastic models.

With these in mind we review the relation between geometric sum and AR(1) models and random walks in section 2 and extend theorem 6.1 of Deshmukh and Teke (2014). In section 3, we review the manifestation of Harris sum in generalized AR(1) models, introduce Harris thinning of random walks where we also generalize theorem 2 of Yannaros (1991) on the Lundberg exponent of p-thinned random walks to the case of Harris thinning. Introducing Harris thinning of renewal processes it is observed that the resulting process is not renewal.

2. Geometric Sums in Stochastic Models

2.1 Geometric Sums and AR(1) Models

The AR(1) model $\{X_n\}$ with i.i.d innovations $\{\epsilon_n\}$ defined by $X_n = \alpha X_{n-1} + \epsilon_n, n \in \mathbb{Z}$, $0 < \alpha < 1$ was generalized to the following two models, among other models, by Lawrance (1978), Gaver and Lewis (1980) and Lawrance and Lewis (1981).

$$X_n = \begin{cases} \alpha X_{n-1} & \text{with probability } \ p, \\ \alpha X_{n-1} + \epsilon_n, & \text{with probability } \ (1-p). \end{cases} \tag{2}$$

$$X_n = \begin{cases} \epsilon_n & \text{with probability } \ p, \\ X_{n-1} + \epsilon_n, & \text{with probability } \ (1-p). \end{cases} \tag{3}$$

Though they noticed the relation of the model to self-decomposable distributions the focus was on the regression and correlation structure of these models. It was Jayakumar and Pillai (1993) who noted the geometric sum nature in the structure of model (2). By assuming stationarity and $X_0 = \epsilon_1$, in terms of CFs (2) is; $\phi_X(t) = p\phi_X(\alpha t) + (1 - p)\phi_X(\alpha t)\phi_X(t)$. That is

$$\phi_X(t) = \frac{p\phi_X(\alpha t)}{1-(1-p)\phi_X(\alpha t)},$$

which suggests invariance of a geometric sum for a given $p \in (0, 1)$. They proved, by virtue of theorem 1.2;

Theorem 2.1 : A sequence $\{X_n\}$ defines the stationary AR(1) scheme (2) with $X_0 = \epsilon_1$, for some $p \in (0, 1)$ iff it is semi-ML.

Corollary 2.2 : A sequence $\{X_n\}$ defines the stationary AR(1) scheme (2) with $X_0 = \epsilon_1$, for every $p \in (0, 1)$ iff it is ML.

Corollary 2.3 : Additionally, if we assume the existence of mean for $\{X_n\}$, then it is exponential.

The discussions till now implicitly assumed that the distributions are non- negative as the focus was on generalizing the model with exponential distribution for X_n. Later Jose and Pillai (1995) showed that the scheme (3) is stationary for each $p \in (0, 1)$ iff X_n is GID. This conclusion follows since under the assumptions and in terms of CFs, (3) is; $\phi_X(t) = p\phi_\epsilon(t) + (1 - p)\phi_X(t)\phi_\epsilon(t)$. That is,

$$\phi_X(t) = \frac{p\phi_\epsilon(t)}{1-(1-p)\phi_\epsilon()},$$

and we have

Theorem 2.4 : A sequence $\{X_n\}$ describes the stationary AR(1) scheme (3) for some $p \in (0,1)$ iff it is the geometric sum of its innovations $\{\epsilon_n\}$. If this structure is required for all $p \in (0,1)$ then $\{X_n\}$ must be GID.

Various ramifications of the models (2) and (3) were studied in Novkovik (1999), Jayakumar and Mathew (2004), Seethalekshmi and Jose (2006), Jose and Thomas (2011) and the references therein.

2.2 Geometric Sums and p-thinning of Random Walks

p-thinning of random walks was introduced by Yannaros (1991). Consider the random walk $\left\{S_n = \sum_{i=1}^{n} X_i\right\}$ where the i.i.d r.v s X_i are the steps of the walk. If we observe every position (n, S_n) with a constant probability p independent of other positions and the random walk, then the observed random walk has steps $Y_i, i = 1, 2, \cdots'$ that are independent copies of the r.v $Y = X_1 + \cdots + X_{\nu_p}$, ν_p is geometric with mean $1/p$. He referred to this relation as: the random walk with steps Y_i is the p-thinning of that with steps X_i and X_i is the p-inverse of Y_i. Invoking Klebanov et al. (1984) he concluded that;

Theorem 2.5 : A random walk has a p-inverse for every $p \in (0,1)$ iff its step distribution is GID.

A random walk is closed under p-thinning if $X_i = cY_i$ for some $c > 0$. Deshmukh and Teke (2014) proved (theorem 6.1) that a random walk with geometrically stable (α-Laplace) step distribution is closed under p-thinning. In fact this is true for the more general semi-α-Laplace steps, as is clear from Pillai (1985) or theorem 1.2 above. The step distribution is geometrically stable only if we demand closure for every $p \in (0,1)$. Invoking theorem 1.2 we have

Theorem 2.6 : A random walk is closed under p-thinning for some $p \in (0,1)$ iff its step distribution is semi-α-Laplace.

Corollary 2.7 : A random walk is closed under p-thinning for every $p \in (0,1)$ iff its step distribution is α-Laplace.

Corollary 2.8 : Additionally, if we assume the existence of second moment for the step distribution, then it is Laplace.

3 Harris Sums in Stochastic Models

3.1 Harris Sums and Generalized AR(1) Models

Harris (1948) proved the invariance of gamma distributions with shape parameter $(1/k)$ under Harris-sum. Satheesh et al. (2002) extended this invariance property to generalized semi-α-Laplace distributions. It is possible that an AR(1) sequence $\{X_n\}$ is composed of k independent AR(1) sequences $\{Y_{n,i}\}, i = 1, \cdots, k$, where $Y_{n,i}$ are identically distributed. That is, for each integer $n > 0$, $X_n = \sum_{i=1}^{k} Y_{n,i}$. For example X_n could be the quantity of water flowing through a river which is the sum of the quantities owing through its k tributaries or the income from sales of a particular item by an agency having k different outlets. Motivated by such possibilities Satheesh et al. (2006) introduced a generalization of the AR(1) scheme (2) as follows.

$$\sum_{i=1}^{k} Y_{n,i} = \begin{cases} \alpha \sum_{i=1}^{k} Y_{n-1,i}, & \text{with probability } \ p, \\ \\ \alpha \sum_{i=1}^{k} Y_{n-1,i} + \sum_{i=1}^{k} \epsilon_{n,i}, & \text{with probability } \ (1-p). \end{cases} \tag{4}$$

Assuming stationarity and $Y_{0,i} = \epsilon_{1,i}$ this reads $\phi(t) = \left\{\frac{p\phi^k(\alpha t)}{1-(1-p)\phi^k(\alpha t)}\right\}^{1/k}$ in terms of CFs. This is invariance of a Harris-sum and invoking theorem 1.4 we have;

Theorem 3.1 : A sequence $\{Y_{n,i}\}$ defines the stationary generalized AR(1) scheme (4) with $Y_{0,i} = \epsilon_{1,i}$, for some $p \in (0,1)$ iff it is generalized semi-α- Laplace.

Corollary 3.2 : A sequence $\{Y_{ni}\}$ defines the stationary generalized AR(1) scheme (4) with $Y_{0,i} = \epsilon_{1,i}$ for every $p \in (0,1)$ iff it is generalized α-Laplace.

Corollary 3.3 : Additionally, if we assume the existence of variance for $Y_{n,i}$, then it is generalized Laplace.

Motivated by similar possibilities Satheesh et al. (2008) considered a generalization of the AR(1) structure (3) as follows.

$$\sum_{i=1}^{k} Y_{n,i} = \begin{cases} \sum_{i=1}^{k} \epsilon_{n,i}, & \text{with probability } \ p, \\ \\ \sum_{i=1}^{k} Y_{n-1,i} + \sum_{i=1}^{k} \epsilon_{n,i}, & \text{with probability } \ (1-p). \end{cases} \tag{5}$$

Assuming stationarity, this reads $\phi_Y(t) = \left\{\frac{p\phi_\epsilon^k(t)}{1-(1-p)\phi_\epsilon^k(t)}\right\}^{1/k}$ in terms of their CFs, which is a Harris-sum of innovations in the model. Thus;

Theorem 3.4 : A sequence $\{Y_{n,i}\}$ defines the stationary generalized AR(1) scheme (5) for some $p \in (0,1)$ iff it is the Harris sum of $\{\epsilon_{n,i}\}$. If it is needed for every $p \in (0,1)$ then $Y_{n,i}$ is HID.

3.2 Harris Sums and Harris Thinning of Random Walks

p-thinning of a random walk is a relation between two r.v s X and Y, where X represents the steps of the original walk and Y that of its p-thinning, given by the geometric-sum

$$Y = X_1 + \cdots + X_{\nu_p} \tag{6}$$

where $X_i, i = 1, 2, \cdots$, are independent copies of X, that is independent of ν_p, a geometric distribution with mean $1/p$. Here we look at the situation where $X = U_1 + \cdots + U_k$ and $Y = V_1 + \cdots + V_k$, where U_j's are independent copies of a r.v U and V_j's are independent copies of a r.v V , $j = 1, \cdots , k$ and the relation between the distributions of U and V. We will refer to U_j's as the component steps of X and V_j's as those of Y. That is, geometric-sum of fixed (k)-sums and the relation between the corresponding component steps (say j-th component step, V_j and U_j , $1 \leq j \leq k$) on both sides of (6). Clearly, this is the Harris-sum

$$V_j = U_1 + \cdots + U_{H_p} \tag{7}$$

of i.i.d r.v s $U_1, U_2, \cdots$, that are independent of the Harris r.v H_p with mean $1/p$ and we call this scheme Harris thinning (H-thinning).

One may think of fluctuations in a financial portfolio where the total return at each step is modeled by the r.v X and the r.v U, the component steps, model the returns from each of the k assets in the portfolio. Then the H-thinned random walk represents the contribution to the total from the j-th asset when the portfolio is observed.

Definition 3.5 : Consider a random walk with steps X_i that are independent copies of a r.v X. Further, let $X_i = U_{i,1} + \cdots + U_{i,k}$ where $U_{i,j}$'s are the component steps ($1 \leq j \leq k$) of X_i, and independent copies of a r.v U_i. Let the random walk $\left\{n, \sum_{i=0}^{n} X_i\right\}$ be observed at stages $n = 1, 2, \cdots$ with probability p independent of other stages and the random walk and not observed with probability $(1 - p)$. Then the random walk whose component step is one of the k component steps in the observed random walk $X_1 + \cdots + X_{\nu_p}$, is the H-thinning of $\left\{\sum_{i=0}^{n} X_i\right\}$.

Theorem 3.6 : A CF $h(t)$ corresponds to the component step distribution of an H-thinned random walk for some $p \in (0,1)$ iff there exists a CF $\phi(t)$ such that

$$h(t) = \left(\frac{p\phi^k(t)}{1-(1-p)\phi^k(t)} \right)^{1/k}. \tag{8}$$

We now look at this as the inverse thinning problem: that is, given a random walk with $h(t)$ as the CF of its component steps, when is

$$\phi(t) = \left(\frac{h^k(t)}{p+(1-p)h^k(t)} \right)^{1/k} \quad \text{for} \quad 0 < p < 1, \quad a\ CF? \tag{9}$$

If $\phi(t)$ is a CF, then we define this relation between the two random walks as:

Definition 3.7 : Given a random walk with $h(t)$ as the CF of the component steps, the random walk with $\phi(t)$ as the CF of the componen steps, as given by (9), is called the H-inverse of the former and the process of deriving it, H-inverse thinning of random walk.

Clearly, a random walk has an H-inverse iff its component step is a Harris- sum. We now have the following characterization invoking the description of HID.

Theorem 3.8 : A random walk has an H-inverse for every $p \in (0,1)$ iff its component step is HID.

Definition 3.9 : A random walk is said to be closed under H-thinning if $\phi(t) = h(ct)$, for some $c > 0$ where $h(t)$ and $\phi(t)$ are CFs of the component steps in (9).

Notice that if a random walk is closed under H-thinning then it is closed under H-inverse thinning as well. Noting that closure under H-thinning is equivalent to invariance under Harris-sum or that $h^k(t)$ is geometrically semi- stable, we invoke theorem 1.4 to conclude,

Theorem 3.10 : A random walk is closed under H-thinning for some $p \in (0,1)$ iff its component step distribution is generalised semi-α-Laplace.

Corollary 3.11 : A random walk is closed under H-thinning for every $p \in (0,1)$ iff its component step distribution is generalised α-Laplace.

Corollary 3.12 : If we assume the existence of second moment for the component steps then it is generalised Laplace.

Next we extend theorem 2 of Yannaros (1991) using the approach in Deshmukh and Teke (2014).

Theorem 3.13 : The Lundberg exponent, if it exists, is the same for a random walk and its H-thinning.

Proof : From (7) the relation between the r.v s representing the component steps is given by the Harris-sum $V = U_1 + \cdots + U_{H_p}$. Assume the existence of the respective moment generating functions $M_V(t), M_U(t)$ and $M_{H_p}(t)$. Using the standard conditioning argument we have $M_V(t) = M_{H_p}(\log M_U(t))$. The Lundberg exponent is the unique positive solution to $M_U(t) = 1$ and let $M_U(\tau) = 1$. Then $M_V(\tau) = M_{H_p}(\log M_U(\tau)) = M_{H_p}(0) = 1$. That proves the assertion.

3.3 Harris Thinning of Renewal Processes

Definition 3.14 : Consider a renewal process, modeling the inter-arrival times T_i of individual items and we are observing batches of k items with a certain probability p and not observing it with probability $(1-p)$. Then the inter-arrival times of the corresponding items (j-th item; $1 \leq j \leq k$) in the observed batches is the H-thinning of the original process.

Remark 3.1 : Notice that the inter-arrival times $T_j, T_{k+j}, T_{2k+j}, \cdots$, of every j-th item in batches of k items themselves do not form a renewal process since they do not appear one after the other in the time domain as a result of the inter-spread of inter-arrival times of other items in the original process. Consequently the H-thinning of a renewal process is not a renewal process.

However, the following result is useful as it relates the inter-arrival times of the corresponding items in the observed batches and those in the original renewal process.

Theorem 3.15 : If $g(\lambda)$ is the LT of the inter-arrival times of the individual items in the observed batches of k items after H-thinning of some renewal process with inter-arrival times having LT $\phi(\lambda)$, then

$$g(\lambda) = \left(\frac{p\phi^k(\lambda)}{1-(1-p)\phi^k(\lambda)} \right)^{1/k}.$$

Consider a service that gets completed at k different stages. Then the above relation between the service times for the different stages of service and those when the service is observed randomly, is useful in understanding the original service time distribution.

4 Concluding Remarks

Discrete analogues of these AR(1) models have been discussed in McKenzie (1986), Bouzar and Jayakumar (2006), Satheesh et al. (2006) and Satheesh et al. (2010). The analogue of change of scale for a discrete r.v X is $c \odot X = Z_1 + \cdots + Z_X$ where

$Z_i, i = 1, 2, \cdots$ are i.i.d as $Z \sim$ Bernoulli(c). For more on this construction see Steutel and van Harn (2004, p.495) and Satheesh and Nair (2002). Motivated by the AR(1) model (3) with X_{n-1} replaced by $cX_{n-1}, c \in [0, 1]$, Kozubowski and Podgorski (2010) introduced random self decomposability which was generalized in Satheesh and Sandhya (2011).

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