International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 12 No. II (December, 2018), pp. 23-27

JUMP OF DISTRIBUTION OF D'_{L^2} THROUGH THE ANALYTIC REPRESENTATION

EGZONA ISENI[1], SHPETIM REXHEPI[2] AND BEDRIJE BEDZETI[3]
[1,2] Mother Theresa University,
Departament of Mathematics, Skopje, Macedonia
[3] University of Tetovo,
Departament of Mathematics, Tetovo, Macedonia

Abstract
In this paper we give some results in the space D'_{L^2}.
We determine the jump of a regular distribution in the space D'_{L^2} in terms of the derivatives of the analytic and the harmonic representation.

1. Introduction

$D_{L^p}, 1 \leq p < \infty$ denotes the space of all infinitely differentiable functions φ for which $\varphi^{(\beta)} \in L^p$ for each n-tuple β of nonnegative integers.

$B = D_{L^\infty}$ is the space of all infinitely differentiable functions which are bounded on $\mathbb{R}^n$.

$\dot{B}$ is the subspace of B that consists of all functions $\varphi \in B$ which vanish at infinity together with each of their derivatives.

A sequence of functions (φ_λ) of D_{L^P} converges to a function φ in the topology of D_{L^P}, $1 \leq p \leq \infty$ as $\lambda \to \lambda_0$ if each $\varphi_\lambda \in D_{L^p}$, $\varphi \in D_{L^p}$, and

----

Key Words : D'_{L^2} *spaces, distribution, Analytic representation, Jump of distribution.*
2010 AMS Subject Classification : Primary 46F20, Secondary 44A15, 46F12.

$$\lim_{\lambda\to\lambda_0} \|\varphi_\lambda^{(\beta)} - \varphi^{(\beta)}\|_{L^p} = \lim_{\lambda\to\lambda_0} \left(\int_{\mathbb{R}^n} |\varphi_\lambda^{(\beta)}(x) - \varphi^{(\beta)}(x)|^p dx \right)^{1/p} = 0 \text{ for every } \beta.$$

A sequence of functions (φ_λ) converges to the function φ in $\dot{B}$ as $\lambda \to \lambda_0$ if each $\varphi_\lambda \in \dot{B}, \varphi \in \dot{B}$, and

$$\lim_{\lambda\to\lambda_0} \|\varphi_\lambda^{(\beta)} - \varphi^{(\beta)}\|_{L^\infty} = 0.$$

D is dense in D_{L^p}, $1 \le p < \infty$ and in $\dot{B}$, but not in $B = D_{L^\infty}$. Also D_{L^p} is dense in L^p. If $\varphi \in D_{L^p}$ for $1 \le p < \infty$ then φ is bounded and converges to 0 at infinity with the same being true for all derivatives of φ.
We have $D \subset D_{L^p} \subset D_{L^q} \subset \dot{B}$ if $1 \le p \le q < \infty$.
D'_{L^p}, $1 \le p < \infty$ is the space of all continuous linear functionals on D_{L^q} where $\frac{1}{p} + \frac{1}{q} = 1$.
$D'_{L'}$ is the space of all continuous linear functionals on $\dot{B}$.
The function $F(z)$ which is analytic on $\mathbb{C} \setminus K$, where K is the support of f, and such that

$$\lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} [F(x+i\epsilon) - F(x-i\epsilon)]\varphi(x)dx = \langle f, \varphi\rangle$$

for all $\varphi \in D$,is called an analytic representation of f.
We say that $U(z)$, harmonic on $Imz > 0$ is a harmonic representation of $f \in D'$ if

$$U(x+iy) \text{ conveges to } f(x) \text{ in } D', \text{ as } y \to 0^+.$$

The following theorem is given in [7].
Theorem 1.1 : If $f \in D'_{L^p}$ then f has Cauchy representation $\hat{f}(z) = \frac{1}{2\pi i}\langle f(t), \frac{1}{t-z}\rangle$, for $Im\ z \ne 0$. A distribution $f \in D'(\mathbb{R})$ is said to have a distributional jump behavior (or jump behavior) at $x = x_0 \in \mathbb{R}$ i.e. $[f]_{x=x_0}$ if it satisfies the distributional asymptotic relation

$$f(x_0 + \epsilon x) = \gamma_- H(-x) + \gamma_+ H(x) + o(1). \tag{1}$$

as $\epsilon \to 0^+$ in D', where $\gamma_\pm$ are constants and H is the Heaviside function and $\gamma_+ - \gamma_- = [f]_{x=x_0}$.
For the Fourier transform of f we write Φ i.e. $\Phi(w) = \int_{-\infty}^{\infty} f(t)e^{-itw}dt$ and for the inverse Fourier transform$(\Phi)^{-1}$.
Let us define the subset of the upper half-plane, $\Delta_\theta^+(x_0)$, to be the set of all z such that $\theta \le arg(z - x_0) \le \pi - \theta$, where $0 < \theta \le \pi/2$. Similarly, we define the subset of the

lower half-plane, $\Delta_\theta^-(x_0)$.

2. Main Results

Theorem 2.1 : Let $T_f \in D'_{L^2}$ be a distribution generated by $f \in L^2$ that have a jump behavior i.e.
$f(x_0+\epsilon x) = \gamma_- H(-x)+\gamma_+ H(x)+o(1)$ as $\epsilon \to 0^+$ at $x = x_0$. Suppose that F is analytic representation of f on $Im\ z \neq 0$.
Then for any $0 < \theta \leq \pi/2$.

$$\lim_{z \to x_0, z \in \Delta\theta^{\pm}(x_0)} (z - x_0)^k F^{(k)}(z) = (-1)^k \frac{(k-1)!}{2\pi i} [f]_{x=x_0}.$$

Note : For the regular distributions $T_f$generated by f we will use the denotation f, i.e.

$$\langle T_f, \varphi\rangle = \langle f, \varphi\rangle = \int_{\mathbb{R}^n} f(t)\varphi(t)dt, \quad \varphi \in D_{L^2}.$$

Proof: Since $\varphi \in D_{L_2}$ and $f \in L^2$, the distribution T_f is well defined.
Using Theorem 8.6.1 in [1] we have that

$$F(z) = \begin{cases} \frac{1}{2\pi}\langle \Phi_+(t), e^{izt}\rangle, & y > 0 \\ \\ -\frac{1}{2\pi}\langle \Phi_-(t), e^{izt}\rangle, & y < 0 \end{cases}$$

is an analytic representation of f.
Differentiating equations (1) we have that

$$\epsilon f'(x_0 + \epsilon x) = \gamma_- H'(-x)(-1) + \gamma_+ H'(x) + o(1)$$

i.e.

$$f'(x_0 + \epsilon x) = \frac{1}{\epsilon}[f]_{x=x_0}\delta(x) + o\left(\frac{1}{\epsilon}\right)$$

since $H'(x) = \delta(x)$ and $\gamma_+ - \gamma_- = [f]_{x=x_0}$.
Differentiating (1) k-times, we have that

$$f^{(k)}(x_0 + \epsilon x) = \frac{1}{\epsilon^k}[f]_{x=x_0}\delta^{(k-1)}(x) + o\left(\frac{1}{\epsilon^k}\right).$$

If we take Fourier transform in the last equation and using the properties

$$\Phi(t - a)(w) = e^{-awi}\Phi(w)$$

$$\Phi(at)(w) = \frac{1}{a}\Phi\left(\frac{w}{a}\right)$$
$$\Phi^{(n)}(t)(w) = (iw)^n\Phi(w)$$

we obtain

$$\Phi^{(k)}(\epsilon x - (-x_0))(\lambda t) = (i\lambda t)^k e^{i\lambda t x_0}\Phi(f)(\lambda t)\frac{1}{\epsilon^k}$$
$$(i\lambda t)^k \frac{1}{\epsilon^k} e^{i\lambda t x_0}\Phi(f)(\lambda t) = \frac{1}{\epsilon^k}[f]_{x=x_0}(i\lambda t)^{(k-1)} \cdot 1 + o(\lambda^{k-1}) \text{ as } \epsilon \to 0^+. \tag{2}$$

Let z belong to $\Delta^{\pm}_{\theta}(x_0)$. Then from equations (1) and (2) we have

$$\begin{aligned} F^{(k)}\left(x_0 + \frac{1}{\lambda}z\right) &= \pm\frac{i^k}{2\pi}\lambda^{k+1}\langle t^K e^{i\lambda x_0 t}\Phi(f_\pm)(\lambda t), e^{izt}\rangle \\ &= \pm\frac{(\pm i)^k}{2\pi}\frac{1}{i}\lambda\ \lambda^{k-1}[f]_{x=x_0}\int_0^\infty t^{k-1}e^{\pm izt}dt + o(\lambda^k) \\ &= \pm\frac{(\pm i)^{k-1}}{2\pi}\lambda^k[f]_{x=x_0}\int_0^\infty t^{k-1}e^{\pm izt}dt + o(\lambda^k). \end{aligned}$$

The last integral is of the form $\int_0^\infty x^{k-1}e^{izx} = (iz)^{-k}\Gamma(k) > 0, Re(k) > 0, Im(z) > 0$, where $\Gamma(k)$ is the Gama function, $\Gamma(k) = (k-1)!$
Now, we obtain

$$\begin{aligned} F^{(k)}\left(x_0 + \frac{1}{\lambda}z\right) &= \pm\frac{(\pm i)^{k-1}}{2\pi}\lambda^k[f]_{x=x_0} iz^{-k}(k-1)! + o(\lambda^k) \\ &= \pm\frac{(\pm i)^k}{2\pi}[f]_{x=x_0}(k-1)!\left(\frac{\lambda}{2}\right)^k + o(\lambda^k) = \\ &\quad \frac{(-1)^k(k-1)!}{2\pi i}[f]_{x=x_0}\left(\frac{\lambda}{z}\right)^k o(\lambda^k) \text{ as } \epsilon \to 0^+. \end{aligned}$$

Put $x_0 + \frac{1}{\lambda}z = z$. When $\lambda \to 0$ then $z \to x_0$, wo we have

$$\lim_{z\to x_0, z\in\Delta\theta^\pm(x_0)}(z-x_0)^k F^{(k)}(z) = (-1)^k\frac{(k-1)!}{2\pi i}[f]_{x=x_0}.$$

Theorem 2.2 : Let $T_f \in D'_{L^2}$ is distribution generated by the differentiable and integrable function $f \in L^2$ that have a jump behavior $f(x_0+\epsilon x) = \gamma_- H(-x) + \gamma_+ H(-x) + o(1)$ as $\epsilon \to 0^+$ at $x = x_0$. Let U be harmonic representation of f , $Imz > 0$ and V is the harmonic conjugate of U. Then

$$\frac{\partial^k U}{\partial x^k}(z) = \frac{(k-1)!}{(-1)^k\pi}[f]_{x=x_0} Im\frac{1}{(z-x_0)^k} + o(|z-x_0|^{-k}),$$

$$\frac{\partial^k V}{\partial x^k}(z) = \frac{(k-1)!}{(-1)^{k+1}\pi}[f]_{x=x_0} Re\frac{1}{(z-x_0)^k} + o(|z-x_0|^{-k}),$$

as $z \to x_0$ on $\Delta_\theta^+(x_0), 0 < \theta \leq \pi/2$.

Proof : Harmonic conjugate differ from each other by constant. We may work with any harmonic representation U of f. Indeed, if U and U_1 are two analytic representations of f, then $U_2 = U - U_1$ represents zero distribution. By the Reflection principle to the real and imaginary parts of U, we have that U_2 admits a harmonic extension to a neighborhood of x_0. Similarly, we get that V_2 admits a harmonic extension to a neighborhood of x_0. Therefore $\frac{\partial^k U_2}{\partial x^k}(z)$, $\frac{\partial^k V_2}{\partial x^k}(z) = O(1)$ in a neighborhood of x_0. We get U and V satisfy the required equations if U_1 and V_1 satisfy the same equation.

Let F be analytic representation of f. Since $U(z) = F(z) - F(\overline{z})$ and $V(z) = i(F(z) + F(\overline{z}))$ for the k-derivative, we have

$$\frac{\partial^k U}{\partial x^k}(z) = F^{(k)}(z) - F^{(k)}(\overline{z}) \text{ and } \frac{\partial^k V}{\partial x^k}(z) = -i(F^{(k)}(z) + F^{(k)}(\overline{z})).$$

From Theorem 2.1. we get the required relations.

References

[1] Bremerman H., Raspredelenija, kompleksnije permenenije I preobrazovanija Furje, Izdatelstvo "Mir" Moskva (1968).

[2] Beltrami E. J., Wohlers M. R., Distributions and the boundary values of analytic functions. Academic Press, New York, (1966).

[3] Carmichael R., Mitrovic D., Distributions and analytic functions, New York, (1989).

[4] Dimovski P., Pilipovic S., Vindas J., New distribution spaces associated to translation-invariant Banach spaces, Monatsh Math 177, (2015), 495-515.

[5] Jantcher L., Distributionen, Walter de Gruyter Berlin, New York, (1971).

[6] Manova-Erakovic V. and Reckovski V., A note on the analytic representations of convergent sequences in S', Filomat, 29:6 (2015), 1419-1424.

[7] Vasko Reckovski, Vesna Manova-Erakovikj, Egzona Iseni, Convergence of sequences of functions in D_{L^p} and $D_{L^p,\eta}, 1 \leq p < \infty$ through their analytic representations, IJMTT, 45:2 (2017), 2231-5373.