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PEBBLING ON UNDIRECTED BIPARTITE GRAPH

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Abstract

A pebbling step of a graph G is consist subtraction of two pebbles from one vertex and placing one to it's an adjacent vertex. The pebbling number of a graph G denoted by $f(G)$, is the least integer m which can move one pebble to target vertex with the help of sequence of pebbling steps. In this paper, we demonstrate the pebbling number of undirected bipartite graph $G = (U, V, E)$.

1. Introduction

A graph $G = (V, E)$ is collection of pair of elements, where V is non-empty set of vertices and E is a set of pairs of elements of V. The numbers of vertices are called order of a graph [1]. Chung was first to introduce pebbling in graphs. Consider a connected graph

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with a fixed number of pebbles which are distributed among all over the vertices. A pebbling step is the subtraction of two pebbles from any arbitrary vertex and addition of one pebble to its adjacent vertex. The pebbling number of vertex v is denoted by $f(G, v)$ in any graph G is the least integer m, which is distributed all over the vertices of graph G and after some sequence of pebbling moves if at least one pebble can sent to v. The pebbling number of graph G is the maximum of $f(G, v)$ over all the vertices of G.

If u is at a distance d from v then $2^d - 1$ pebbles on u are not sufficient to sent one pebble to v from u. Results of graph pebbling [2, 3, 4, 5].

1. $f(G) \geq \max\{|V(G)|, 2^D\}$, where D is the diameter of graph G.
2. $f(K_n) = n$, where K_n is the complete graph.
3. $f(P_n) = 2^{n-1}$, where P_n is the path graph with n vertices.

A undirected bipartite graph is a simple graph in which the set of vertices can be partitioned into two sets U and V such that every edge is between a vertex of U and a vertex of V. It is represented as $G = (U, V, E)$ [1].

In an undirected bipartite graph $G = (U, V, E), s = |U|+|V|$ that is the sum of numbers of all vertices in non-empty sets U and V of G.

In general $G = (U, V, E), |U| = \{u_1, u_2, \cdots, u_m\} = m$ number of vertices and $|V| = \{v_1, v_2, \cdots, v_n\}$ n number of vertices.

Consider k = largest path L of undirected bipartite graphs containing k vertices of $|U| + |V|$ and $\overline{k}$ = remaining numbers of vertices in G other than k contains at each vertex at least one pebble.

2. Pebbling on Undirected Bipartite Graph

In this section we find and demonstrate the pebbling number of undirected bipartite graph.

Lemma 2.1 [7] : Let $P_n = v_1, v_2, \cdots, v_n$ be a path. Then P_n has weight $\sum_{i=1}^{n-1} 2^{i-1}p(v_i)$ with respect to v_n and this is written as $wp_n(v_n)$.

Lemma 2.2 [7] : Let $P_n = v_1, v_2, \cdots, v_n$ be a path. If $wp_n(v_n) \geq k.2^{n-1}$, then at least k pebbles can be moved from p_n/v_n to v_n.

2.3 Main Results

Main Theorem 2.3.1 : Let $G = (U, V, E)$ be a undirected bipartite graph and if there is at least one path containing every vertices of $|U|$ and $|V|$ of G then,

$$f(G) = 2^{s-1}, \quad \text{where} \quad s = |U| + |V|.$$

Proof : We use mathematical induction on s.

Basic of Induction : When $s = 2$ (i.e. $|U| = 1$ and $|V| = 1$) then without loss of generality, result is obvious. So, one pebble is sent to target vertex u_1.

Induction Hypothesis : Let us assume that result holds true for s, i.e. $f(G) = 2^{s-1}$.

$$\text{Target vertex} = \{u_m\}$$

$$|U| = \{u_1, u_2, \cdots, u_m\}$$

$$|V| = \{v_1, v_2, \cdots, v_n\}.$$

So, we can say $f(G) = 2^{m+n-1}, s = m + n$.

Induction Step : To show induction on $(s + 1)$ holds.

Such that $f(G) = 2^{s+1-1} = 2^s$ can sent one pebble to target vertex.

Since in induction step on $(s + 1), f(G) = 2^{m+1+n-1} = 2^{m+n}$.

Let $L_1 = \{u_1, v_1, u_2, v_2, \cdots, u_{m-2}, v_{n-2}, u_{m-1}, v_{n-1}, u_m\}$ and $L_2 = \{u_m, v_n, u_{m+1}\}$ be any two subpaths of undirected bipartite graph G [7], where $G = L_1 \cup L_2$.

As we know, $f(P_n) = 2^{n-1}$, where P_n is the path graph with n vertices.

Case (a) : In $L_1 = \{u_1, v_1, u_2, v_2, \cdots, u_{m-2}, v_{n-2}, u_{m-1}, v_{n-1}, u_m\}$, there is a largest path of $m + 1 + n - 1 - 1 = m + n - 1$ vertices.

Since in induction hypothesis, result holds for $s = m + n$. It is obvious, with the help of induction hypothesis, we can pebble the target vertex (u_m).

$$f(L_1) = 2^{m+n-2} \tag{2.3.1.1}$$

Case (b) : In $L_2 = \{u_m, v_n, u_{m+1}\}$, the worst case is $p(u_m) = 0$ and all pebbles are on u_{m+1} vertex then,

$$\begin{aligned} f(L_2) &= 2^{m+1+n-1-(m+1-2+n-1)} \\ &= 2^2 \end{aligned} \tag{2.3.1.2}$$

We can pebble the target vertex u_m.

Then from equation (2.3.1.1) and (2.3.1.2), we get,

$$f(G) = 2^{m+n-2} \cdot 2^2 = 2^{m+n}$$

So, u_m can be pebbled with one pebble.

Corollary 2.3.2 : In $G = (U, V, E)$ undirected bipartite graph where $|U| = \{u_1, u_2, \cdots, u_m, u_{m+1}\}$ and $|V| = \{v_1, v_2, \cdots, v_n\}$ in figure 2.

Let $p(u_m) = 2^2$ via L_2 (when target vertex is u_{m+1}) and $p(u_m) = 2^{m+n-2}$ via L_1 (when target vertex is u_1). Then using lemma 2.1 and lemma 2.2 we can say, [7]

$$w_G(u_m) \geq 2^{m+n} = 2^s.$$

Illustration 2.3.3 : Let $G = (U, V, E)$ be a undirected bipartite graph, where

$$|U| = \{a, c, e, g\} \text{ and } |V| = \{b, d, f\}.$$

Then, $f(G) = 2^{4+3-1} = 2^6$.

Main Theorem 2.3.4 : If in $G = (U, V, E)$ undirected bipartite graph, there is not a path which contains all the vertices of G then, for $0 \leq \overline{k} \leq m, n$,

$$f(G) = 2^{k-1} + \overline{k}.$$

Proof : There are two possibilities,

Case (a) : When $\overline{k} = 0$, then according to statement, $f(G) = 2^{k-1} + 0 = 2^{k-1}$. Then, using theorem 2.3.1 result holds true.

Case (b): When $1 \leq \overline{k} \leq m, n$, then $f(G) = 2^{k-1} + \overline{k}$.

Using mathematical induction on $\overline{k}$.

Basic of Induction : When $\overline{k} = 1$, then in L path, one pebble is sent to target vertex whether the remaining (one vertex with one pebble) is connected to L path or not.

Induction Hypothesis : Let us assume that result holds true for $\overline{k}$. So, for $\overline{k}$ we can pebble the target vertex.

Induction Step : To prove induction holds for $(\overline{k}+1)$. i.e.

$$f(G) = 2^{k-1} + (\overline{k}+1)$$

$$\text{Or} \quad f(G) = (2^{k-1} + \overline{k}) + 1 \quad \text{or} \quad f(G) = (2^{k-1} + 1) + \overline{k}.$$

Using $\overline{k}$ on induction hypothesis and for 1 on the basic of induction, we can sent one pebble to target vertex.

3. Concluding Remark

In this paper, we calculate the pebbling number of undirected bipartite graph and its various cases related to undirected bipartite graph using the properties of path graph P_n. It is widely used in the field of computational programming as well as it represents the data preferences. Also, undirected bipartite graphs are abetment to problems of object recognition and in diverse fields including cloud computing.

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