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# SOME NEW RESULTS ON CHROMATIC TRANSVERSAL DOMINATION IN GRAPHS 

S. K. VAIDYA ${ }^{1}$ AND A. D. PARMAR ${ }^{2}$<br>${ }^{1}$ Department of Mathematics, Saurashtra University, Rajkot - 360005, Gujarat, India<br>${ }^{2}$ Atmiya Institute of Technology \& Science for Diploma Studies, Rajkot - 360005, Gujarat, India


#### Abstract

A vertex dominating set $D$ of $V(G)$ is called a chromatic transversal dominating set of $G$ if $D$ intersects every color class of $G$. The minimum cardinality of $D$ is called a chromatic transversal domination number of $G$. In this work we contribute some new results on chromatic transversal domination.


## 1. Introduction

We consider simple, finite, undirected and connected graph $G=(V(G), E(G))$. We denote the degree of a vertex $v$ in a graph $G$ by $d_{G}(v)$. The maximum degree among the vertices of $G$ is denoted by $\Delta(G)$. For any real number $n,\lceil n\rceil$ denotes the smallest integer not less than $n$ and $\lfloor n\rfloor$ denotes the greatest integer not greater than $n$. For the various graph theoretic notation and terminology we follow West [7]. For standard

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terminology and terms related to coloring are used in the sense of Zhang [9] while for any undefined terms related to the concept of domination we refer to Haynes et al. [1]. The study of graph coloring and its related concepts are getting momentum due to its diversified applications for the solution of many real life problems such as scheduling time-table, compiler register allocation, assigning mobile and radio frequency, etc. An excellent discussion on theory of graph coloring is carried out by Zhang [9].
An independent set of vertices in a graph $G$ is a set of pairwise non-adjacent vertices of $G$. A proper $k$ - coloring of a graph $G$ is a function $f: V(G) \rightarrow\{1,2, \cdots, k\}$ such that $f(u) \neq f(v)$ for all $u v \in E(G)$. The color class $S_{i}$ is the subset of vertices of $G$ that is assigned to color $i$. The chromatic number $\chi(G)$ is the minimum number $k$ for which $G$ admits proper $k$ - coloring. Equivalently the chromatic number $\chi(G)$ of a graph $G$ is defined to be the minimum number of colors required to color the vertices of a graph $G$ in such a way that no two adjacent vertices of $G$ receive the same color. The minimum $k$ such that we can partition $V(G)=S_{1} \cup S_{2} \cup \cdots \cup S_{k}$, where each $S_{i}$ is independent set, is the chromatic number $\chi(G)$. A partition of $V(G)$ in to $\chi(G)$ independent sets is called $\chi$ - partition of $G$.

The set $D \subseteq V(G)$ of vertices in a graph $G$ is called dominating set if every vertex $v \in V(G)$ is either an element of $D$ or is adjacent to an element of $D$. The minimum cardinality of a dominating set is called the domination number of $G$ which is denoted by $\gamma(G)$.
The domination in graph is one of the fastest growing concept in graph theory. Many variants of domination models are available in literature: independent domination, total domination, equitable domination, total equitable domination are among worth to mention. Independent sets play a significant role in graph theory in general. They appear in theory of trees, coloring of graphs and matching theory.
If $\mathcal{C}=\left\{S_{1}, S_{2}, \cdots, S_{k}\right\}$ is a $k$ - coloring of a graph G then a subset $D$ of $V(G)$ is called a transversal of $C$ if $D \cap S_{i} \neq \phi$ for all $i \in\{1,2, \cdots, k\}$. A dominating set $D$ of a graph $G$ is called a chromatic transversal dominating set (cdt - set) of $G$ if $D$ is transversal of every chromatic partition of $G$. The minimum cardinality of a cdt - set $D$ of $G$ is called the chromatic transversal domination number of $G$ and is denoted by $\gamma_{c t}(G)$. This concept was introduced by Michaelraj et al. [4].
Definition 1.1: The corona $G \circ H$ of two graphs $G$ and $H$ is defined to be the graph
obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$ and joining $i$-th vertex of $G$ with an edge to every vertex in the ith copy of $H$.
Definition 1.2: The crown $C r_{n}$ is $C_{n} \circ K_{1}$ is obtained by joining a pendant edge to each vertex of $C_{n}$.
Definition 1.3: The cartesian product of $G$ and $H$ is a graph, denoted as $G \times H$, whose vertex set $V(G) \times V(H)$. Two vertices $(g, h)$ and $\left(g^{\prime}, h^{\prime}\right)$ are adjacent precisely if $g=g^{\prime}$ and $h h^{\prime} \in E(H)$, or $g g^{\prime} \in E(G)$ and $h=h^{\prime}$. Thus, $V(G \times H)=\{(g, h)=g \in V(G)$ and $h \in V(H)\}$ and $E(G \times H)=\left\{(g, h)\left(g^{\prime}, h.\right)=g=g^{\prime}, h h^{\prime} \in E(H)\right.$ or $\left.g g^{\prime} \in E(G), h=h^{\prime}\right\}$.
Definition 1.4: The book $B_{n}$ is a graph $K_{1, n} \times P_{2}$.
Definition 1.5: The square of a graph $G$ denoted by $G_{2}$ has the same vertex set as of $G$ and two vertices are adjacent in $G^{2}$ if they are at distance of 1 or 2 apart in $G$.
Definition 1.6: The switching of a vertex $v$ of $G$ means removing all the edges incident to $v$ and adding edges joining $v$ to every vertex which is not adjacent to $v$ in $G$. We denote the resultant graph by $\tilde{G}$.
In this paper we obtained the chromatic transversal domination number of some graphs.
Definition 1.7. [2] : For a bipartite graph $G$, $\gamma_{c t}(G)=\gamma(G)$ or $\gamma_{c t}(G)=\gamma(G)+1$. All bipartite graphs for which $\gamma_{c t}(G)=\gamma(G)$ are called type I graphs. Other graphs are type II graphs.
Proposition 1.8 [2]: Let $G$ and $H$ be any graphs. Then

$$
\gamma_{c t}(G \circ H)=\left\{\begin{array}{ll}
|V(G)| ; . & \text { if } \chi(G)>\chi(H) \\
|V(G)|+\chi(H)-\chi(G) ; & \text { otherwise }
\end{array} .\right.
$$

Proposition 1.9 [8]: If $\tilde{C}_{n}$ is the graph obtained by an arbitrary vertex $v$ in cycle $C_{n}$ then

$$
\chi\left(\tilde{C}_{n}\right)= \begin{cases}2 & \text { if } n=4 \\ 3 ; & \text { if } n>4\end{cases}
$$

Proposition 1.10 [5]: If $\tilde{C}_{n}$ is the graph obtained by switching of an arbitrary vertex $v$ in cycle $C_{n}$ then

$$
\chi\left(\tilde{C}_{n}\right)= \begin{cases}l & \text { if } n=4 \\ 2 ; & \text { if } n=5 \\ 3 ; & \text { if } n \geq 6\end{cases}
$$

Proposition 1.11 [3] : Let $B_{n}$ be any book graph. Then $\gamma\left(B_{n}\right)=2$, where $n \geq 3$.
Proposition 1.12 [6] : Let $G$ be a graph with $\gamma(G) \leq \chi(G)$. Then $\gamma_{c t}(G)=\chi(G)$.

## 2. Main Results

Theorem 2.1: Let $G$ be a graph with $\chi(G) \leq 2$. Then $\gamma_{c t}(G)=\gamma(G)$, where $G$ is not type II graph.

Proof : Let $G$ be a graph with $\chi(G) \leq 2$ which is not type II. To prove the result, we consider following two cases:

Case I : If $\chi(G)=1$ then $G$ is null graph. Therefore $\gamma_{c t}(G)=\gamma(G)$.
Case II : If $\chi(G)=2$ then $G$ is bipartite graph. Moreover the graph $G$ is not type II. Hence by Definition 1.7, $\gamma_{c t}(G)=\gamma(G)$.

Theorem 2.2: If $G$ is a connected graph of order $n>2$ with $\chi(G)>2$ and $G^{\prime}$ is a graph obtained by duplication of every vertex of a connected graph $G$ by an edge then $\gamma_{c t}\left(G^{\prime}\right)=n$.

Proof : If $G$ is a connected graph of order $n>2$ with $\chi(G)>2$ and $G^{\prime}$ is a graph obtained by duplication of every vertex of a connected graph $G$ by an edge. Then $G^{\prime} \cong G \circ P_{2}$. We know that $\gamma_{c t}(G)>2$ as $\chi(G)>2$ and $\gamma_{c t}\left(P_{2}\right)=2$. Hence by Proposition $1.8 \gamma_{c t}\left(G^{\prime}\right)=n$ as $\chi(G)>\chi\left(P_{2}\right)$.

Theorem 2.3: For the crown $C r_{n}, \gamma_{c t}\left(C r_{n}\right)=n$.
Proof : Let $G$ be a cycle with $n$ vertices and $H$ be $K_{1}$. Then $C r_{n} \cong C_{n} \circ K_{1}$. Moreover $\chi\left(C_{n}\right)=2$ for $n$ is even, $\chi\left(C_{n}\right)=3$ for $n$ is odd and $\chi\left(K_{1}\right)=1$. Therefore $\chi\left(C_{n}\right)>\chi\left(K_{1}\right)$. Hence by Proposition 1.8, $\gamma_{c t}\left(C r_{n}\right)=n$.

Lemma 2.4 : For the book $B_{n}, \chi\left(B_{n}\right)=2$, for all $n>4$.
Proof : Let $B_{n}=K_{1, n} \times P_{2}$ be a book. Thus from the definition of $B_{n}, B_{n}$ is a bipartite as $B_{n}$ does not contains any odd cycle. Hence $\chi\left(B_{n}\right)=2$.

Theorem 2.5 : . For any book $B_{n}, \gamma_{c t}\left(B_{n}\right)=2$, for all $n>4$.
Proof : Let $B_{n}=K_{1, n} \times P_{2}$ be a book. Now by Lemma 2.4, $\chi\left(B_{n}\right)=2$ and by Proposition 1.11, $\gamma\left(B_{n}\right)=2$ for all $n>4$. Therefore $\chi\left(B_{n}\right)=\gamma\left(B_{n}\right)=2$. Hence, $\gamma_{c t}\left(B_{n}\right)=\chi\left(B_{n}\right)=2$, for all $n>4$.

## Lemma 2.6 :

$$
\chi\left(C_{n}^{2}\right)= \begin{cases}3 ; & \text { if } n \equiv 0(\bmod 3) \\ 4 ; & \text { if } n \equiv 1(\bmod 3) \\ 5 ; & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

Proof: Let $V\left(C_{n}\right)=V\left(C_{n}^{2}\right)=\left\{v_{1}, v_{2}, v_{3}, \cdots, v_{n}\right\}$ be the vertex set where $d_{C_{n}^{2}}\left(v_{i}\right)=4$ for all $i \in\{1,2,3, \cdots, n\}$.
Moreover by definition of $C_{n}^{2}, K_{3}$ is a subgraph of $C_{n}^{2}$. Therefore number of independent sets are atleast three. To prove the result we consider the following cases:

Case I : If $n \equiv 0(\bmod 3)$.
Now we construct different sets of vertices ad follows:

$$
\begin{aligned}
S_{1} & =\left\{v_{3 i+1} / 0 \leq i \leq \frac{n}{3}-1\right\} \\
S_{2} & =\left\{v_{3 i+2} / 0 \leq i \leq \frac{n}{3}-1\right\} \\
S_{3} & =\left\{v_{3 i} / 0 \leq i \leq \frac{n}{3}\right\}
\end{aligned}
$$

Further $V\left(C_{n}^{2}\right)=S_{1} \cup S_{2} \cup S_{3}$, where each $S_{i}$ is a minimum independent set as $K_{3}$ is a subgraph of $C_{2 n}$. Hence $\chi\left(C_{n}^{2}\right)=3$ if $n \equiv 0(\bmod 3)$.
Case II : If $n \equiv 1(\bmod 3)$.
In this case we construct different sets of vertices ad follows:

$$
\begin{aligned}
S_{1} & =\left\{v_{3 i+1} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor-1\right\} \\
S_{2} & =\left\{v_{3 i+2} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor-1\right\} \\
S_{3} & =\left\{v_{3 i} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor\right\} \\
S_{4} & =\left\{v_{n}\right\}
\end{aligned}
$$

Further $V\left(C_{n}^{2}\right)=S 1 \cup S_{2} \cup S_{3} \cup S_{4}$, where each $S_{i}$ is a minimum independent set as $K_{3}$ is a subgraph of $C 2_{n}$ and there exists a vertex $u \in S_{i}$ for all $i \in\{1,2,3\}$ such that $u v_{n} \in E\left(C_{n}^{2}\right)$. Hence $\chi\left(C_{n}^{2}\right)=4$ if $n \equiv 1(\bmod 3)$.

Case III : If $n \equiv 2(\bmod 3)$. Here we construct different sets of vertices as follows:

$$
\begin{aligned}
S_{1} & =\left\{v_{3 i+1} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor-1\right\} \\
S_{2} & =\left\{v_{3 i+2} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor-1\right\} \\
S_{3} & =\left\{v_{3 i} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor\right\} \\
S_{4} & =\left\{v_{n-1}\right\} \\
S_{5} & =\left\{v_{n}\right\} .
\end{aligned}
$$

Further $V\left(C_{n}^{2}\right)=S_{1} \cup S_{2} \cup S_{3} \cup S_{4} \cup S_{5}$, where each $S_{i}$ is a minimum independent set as $K_{3}$ is a subgraph of $C_{n}^{2}$ and there exists a vertex $u \in S_{i}$ for all $i \in\{1,2,3\}$ such that $u v_{n-1} \in E\left(C_{n}^{2}\right)$ and $u v_{n} \in E\left(C_{n}^{2}\right)$. Hence $\chi\left(C_{n}^{2}\right)=5$ if $n \equiv 2(\bmod 3)$.
Theorem 2.7:

$$
\gamma_{c t}\left(C_{n}^{2}\right)= \begin{cases}\left\lceil\frac{n}{5}\right\rceil ; & \text { for } n \equiv 0(\bmod 3) \text { and } n \geq 12 \\ \left\lceil\frac{n}{5}\right\rceil+1 ; & \text { for } n \equiv 1(\bmod 3) \text { and } n \geq 13 \\ \left\lceil\frac{n}{5}\right\rceil+2 ; & \text { for } n \equiv 2(\bmod 3) \text { and } n \geq 14\end{cases}
$$

Proof: If $D$ is any color transversal dominating set of $C_{n}^{2}$ then without loss of generality $v_{1} \in D$ as $d_{C_{n}^{2}}\left(v_{i}\right)=4$ for all $i=\{1,2, \cdots, n\}$. To prove the result we consider the following cases:
Case I: For $n \equiv 0(\bmod 3)$.
In this case from the Lemma 2.6,

$$
\begin{aligned}
S_{1} & =\left\{v_{3 i+1} / 0 \leq i \leq \frac{n}{3}-1\right\} \\
S_{2} & =\left\{v_{3 i+2} / 0 \leq i \leq \frac{n}{3}-1\right\} \\
S_{3} & =\left\{v_{3 i} / 0 \leq i \leq \frac{n}{3}\right\}
\end{aligned}
$$

are three minimum number of independent sets of vertices with color 1,2 and 3 respectively.
Now we construct a set $D=\left\{V_{5 i+1} / 0 \leq i \leq\left\lceil\frac{n}{5}\right\rceil\right\}$. Then $|D|=\left\lfloor\frac{n}{5}\right\rfloor$. Moreover $D$ is a chromatic transversal dominating set of $C_{n}^{2}$ as $D \cap S_{i} \neq \phi$ for all $i$. Further we claim that $|D|$ is a minimum because for any $u \in D, D-\{u\}$ is not a color transversal dominating set of $C_{n}^{2}$ as $(D-\{u\}) \cap S_{i}=\phi$ for some $i$. Therefore containing the vertices
less than that of $|D|$ can not be a chromatic transversal dominating set of $C_{n}^{2}$. Hence $\gamma_{c t}\left(C_{n}^{2}\right)=\left\lceil\frac{n}{5}\right\rceil$.
Case II : For $n \equiv 1(\bmod 3)$.
In this case from the Lemma 2.6,

$$
\begin{aligned}
S_{1} & =\left\{v_{3 i+1} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor-1\right\} \\
S_{2} & =\left\{v_{3 i+2} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor-1\right\} \\
S_{3} & =\left\{v_{3 i} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor\right\} \\
S_{4} & =\left\{v_{n}\right\}
\end{aligned}
$$

are minimum number of independent sets of vertices with color $1,2,3$ and 4 respectively. Now we construct a set $D=\left\{V_{5 i+1} / 0 i \leq\left\lfloor\frac{n}{5}\right\rfloor\right\} \cup\left\{v_{n}\right\}$. Then $|D|=\left\lceil\frac{n}{5}\right\rceil+1$. Moreover $D$ is a chromatic transversal dominating set of $C_{n}^{2}$ as $D \cap S_{i} \neq \phi$ or all $i$. Further we claim that $|D|$ is a minimum because for any $u \in D, D-\{u\}$ is not a color transversal dominating set of $C_{n}^{2}$ as $(D-\{u\}) \cap S_{i}=\phi$ for some $i$. Therefore containing the vertices less than that of $|D|$ can not be a chromatic transversal dominating set of $C_{n}^{2}$. Hence $\gamma_{c t}\left(C_{n}^{2}\right)=\left\lceil\frac{n}{5}\right\rceil+1$.
Case III : For $n \equiv 1(\bmod 3)$.
For case from the Lemma 2.6,

$$
\begin{aligned}
S_{1} & =\left\{v_{3 i+1} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor-1\right\} \\
S_{2} & =\left\{v_{3 i+2} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor-1\right\} \\
S_{3} & =\left\{v_{3 i} / 0 \leq i \leq\left\lfloor\frac{n}{3}\right\rfloor\right\} \\
S_{4} & =\left\{v_{n-1}\right\} \\
S_{5} & =\left\{v_{n}\right\}
\end{aligned}
$$

are minimum number of independent sets of vertices with color $1,2,3,4$ and 5 respectively.
Now we construct a set $D=\left\{V_{5 i+1} / 0 \leq i \leq\left\lfloor\frac{n}{5}\right\rfloor\right\} \cup\left\{v_{n-1}, v_{n}\right\}$. Then $|D|=\left\lfloor\frac{n}{5}\right\rfloor+2$. Moreover $D$ is a chromatic transversal dominating set of $C_{n}^{2}$ as $D \cap S_{i} \neq \phi$ for all $i$. Further we claim that $|D|$ is a minimum because for any $u \in D, D-\{u\}$ is not a color transversal dominating set of $C_{n}^{2}$ as $(D-\{u\}) \cap S_{i}=\phi$ for some $i$. Therefore containing
the vertices less than that of $|D|$ can not be a chromatic transversal dominating set of $C_{n}^{2}$. Hence $\gamma_{c t}\left(C_{n}^{2}\right)=\left\lceil\frac{n}{5}\right\rceil+2$.
Theorem 2.8 : Let $\tilde{C}_{n}$ be the graph obtained by switching of an arbitrary vertex $v$ in cycle $C_{n}$ then

$$
\gamma_{c t}\left(\tilde{C}_{n}\right)= \begin{cases}2 ; & \text { if } n=4 \\ 3 ; & \text { if } n>4\end{cases}
$$

Proof: Let $\tilde{C_{n}}$ be the graph obtained by switching of an arbitrary vertex $v$ in cycle $C_{n}$. Now by Proposition 1.10 and 1.9, $\gamma\left(\tilde{C}_{n}\right) \leq \chi\left(\tilde{C}_{n}\right)$. Therefore $\gamma_{c t}\left(\tilde{C}_{N}\right)=\chi\left(\tilde{C}_{n}\right)$. Hence

$$
\gamma_{c t}\left(\tilde{C}_{n}\right)= \begin{cases}2 ; & \text { if } n=4 \\ 3 ; & \text { if } n>4\end{cases}
$$

## 3. Concluding Remarks

The concept of chromatic transversal dominating set is interesting as it relates two important concepts of graph theory, namely graph coloring and domination in graph. We have investigated this parameter for crown, book graph, $C_{n}^{2}$ and $\tilde{C}_{n}$. It has been shown that the graphs for which chromatic transversal domination number is equal to domination number.

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