International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 9 No. IV (December, 2015), pp. 81-91

UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE ONE SMALL FUNCTION WITH ITS DIFFERENTIAL POLYNOMIAL

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Abstract

In this paper, we investigate the uniqueness problems of meromorphic functions that share a small function with its differential polynomials, and give a result which is related to a conjecture of R. Bruck and improve the result of Lipei Liu [11].

1. Introduction and Main Results

In this paper, meromorphic functions mean meromorphic in the complex plane. We use the standard notations of Nevanlinna theory, which can be found in [10]. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r,a) = S(r,f)$, i.e. $T(r,a) = o(T(r,f))$ as $r \to +\infty$ possibly outside a set of finite linear measure. We say

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Key Words : *Uniqueness, Meromorphic function, Differential polynomial, Shared value, Small function.*
2010 AMS Subject Classification : Primary 30D35.

that two meromorphic functions f and g share a small function a IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

For any constant a, we denote by $N_{k)}(r,\frac{1}{f-a})$ the counting function for zeros of $f-a$ with multiplicity no more than k, and by $\overline{N}_{k)}(r,\frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,\frac{1}{f-a})$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r,\frac{1}{f-a})$ be the corresponding one for which multiplicity is not counted. Set

$N_k(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})+\overline{N}_{(2}(r,\frac{1}{f-a})+...+\overline{N}_{(k}(r,\frac{1}{f-a})$.

We define

$$\Theta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)} \quad , \quad \delta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)}.$$

We further define

$$\delta_k(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{N_k(r,\frac{1}{f-a})}{T(r,f)}.$$

Obviously,

$$1\geq\Theta(a,f)\geq\delta_k(a,f)\geq\delta(a,f)\geq 0.$$

In additional, we shall also use the following notations:

Let f and g be two non-constant meromorphic functions such that f and g share 1 IM. We denote by $\overline{N}_L(r,\frac{1}{f-1})$ the counting function for 1-point of both f and g about which f has larger multiplicity than g, with multiplicity being not counted, and denote by $N_{11}(r,\frac{1}{f-1})$ the counting function for common simple 1-point of both f and g, and denote by $N_{(22}(r,\frac{1}{f-1})$ the counting function of those same multiplicity 1-point of both f and g and the multiplicity is ≥ 2. In the same way, we can define $\overline{N}_L(r,\frac{1}{g-1})$, $N_{11}(r,\frac{1}{g-1})$, and $N_{(22}(r,\frac{1}{g-1})$. Especially, if f and g share 1 CM, then $\overline{N}_L(r,\frac{1}{g-1})=0$.

Bruck considered the uniqueness problem of an entire function sharing one value with its derivative and proved the following result.

Theorem A([1]) : Let $f(z)$ be a non-constant entire function. If f and f' share the value 1 CM and if $N(r,\frac{1}{f'})=S(r,f)$. Then $\frac{f'-1}{f-1}\equiv c$ for some constant $c\in\mathbb{C}\setminus\{0\}$.

Yang[6] and Zhang[8] extended Theorem A and obtained many excellent theorems.

Theorem B([6]) : Let $f(z)$ be a non-constant meromorphic function and k be a positive integer. Suppose that f and $f^{(k)}$ share 1 CM and

$2\overline{N}(r,f)+\overline{N}(r,\frac{1}{f'})+N(r,\frac{1}{f^{(k)}})<(\lambda+o(1))T(r,f^{(k)})$,
for $r\in I$, where I is a set of infinite linear measure and λ satisfies $0<\lambda<1$, then $\frac{f^{(k)}-1}{f-1}\equiv c$ for some constant $c\in\mathbb{C}\setminus\{0\}$.

Yu considered the problem of a meromorphic functions sharing one small function with its derivative and proved the following theorem.

Theorem C([7]) : Let f be a non-constant meromorphic function and $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. If

(i) f and a have no common poles,

(ii) $f-a$ and $f^{(k)}-a$ share the value 0 CM,

(iii) $4\delta(0,f)+2(8+k)\Theta(\infty,f)>19+2k$, then $f\equiv f^{(k)}$, where k is a positive integer.

In the same paper, Yu posed four open questions. Lahiri [2], Liu and Gu [3], and Zhang [9] studied this questions and obtained a series results which answered the open questions. Recently, Zhang and Lu[10] considered the problem of f^n and $f^{(k)}$ sharing a small function and obtained the following results, which are the improvements and complements of above theorems.

Theorem D([10]) : Let $k(\geq 1)$,$n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also, let $a=a(z)(\not\equiv 0,\infty)$ be a small function such that $T(r,a)=S(r,f)$, as $r\to\infty$. Suppose that f^n and $f^{(k)}$ share a IM and

$$4\overline{N}(r,f)+2\overline{N}(r,\frac{1}{(f^n/a)'})+\overline{N}(r,\frac{1}{f^{(k)}})+2N_2(r,\frac{1}{f^{(k)}})<(\lambda+o(1))T(r,f^{(k)})$$

or f^n and $f^{(k)}$ share a CM and

$$2\overline{N}(r,f)+2\overline{N}(r,\frac{1}{(f^n/a)'})+N_2(r,\frac{1}{f^{(k)}})<(\lambda+o(1))T(r,f^{(k)}),$$

for $r\in I$, where I is a set of infinite linear measure and λ satisfies $0<\lambda<1$ then $\frac{f^{(k)}-a}{f-a}\equiv c$ for some constant $c\in\mathbb{C}\setminus\{0\}$.

Theorem E([10]) : Let $k(\geq 1)$,$n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also, let $a=a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. If f^n and $f^{(k)}$ share a IM and

$$(2k+6)\Theta(\infty,f)+4\Theta(0,f)+2\delta_{2+k}(0,f)>2k+12-n,$$

or f^n and $f^{(k)}$ share a CM and

$$(k+3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{2+k}(0, f) > k + 6 - n,$$

then $f^n = f^{(k)}$.

In 2010, Lipei Lu [11] improved the above result and proved the following theorem.

Theorem F([11]) : Let $k(\geq 1)$,$n(\geq 1)$, $m(\geq 2)$ be integers and f be a non-constant meromorphic function. Also, let $a = a(z)(\not\equiv 0, \infty)$ be a small function such that $T(r, a) = S(r, f)$, as $r \to \infty$. Suppose that f^n and $(f^{(k)})^m$ share a IM and

$$\frac{4}{m}\overline{N}(r, f) + \frac{2}{m}\overline{N}(r, \frac{1}{(f^n/a)'}) + \frac{5}{m}\overline{N}(r, \frac{1}{f^{(k)}}) < (\lambda + o(1))T(r, f^{(k)}),$$

or f^n and $(f^{(k)})^m$ share a CM and

$$\frac{2}{m}\overline{N}(r, f) + \frac{1}{m}\overline{N}(r, \frac{1}{(f^n/a)'}) + \frac{2}{m}\overline{N}(r, \frac{1}{f^{(k)}}) < (\lambda + o(1))T(r, f^{(k)}),$$

for $r \in I$, where I is a set of infinite linear measure and λ satisfies $0 < \lambda < 1$, then $\frac{(f^{(k)})^m - a}{f^n - a} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

We need the following definition.

Definition 1([14]) : Let $n_{0j}, n_{1j}, ..., n_{kj}$ be non negative integers.

- The expression $M_j[f] = (f)^{n_{0j}}(f')^{n_{1j}}...(f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d(M_j) = \sum\limits_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum\limits_{i=0}^{k}(i+1)n_{ij}$.
- The sum $P[f] = \sum\limits_{i=1}^{t} b_j M_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P) = max\{d(M_j) : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, ...t$.
- The number $\underline{d}(P) = min\{d(M_j) : 1 \leq j \leq t\}$ and k the highest order of the derivative of f in $P[f]$ are called respectively the lower degree and order of $P[f]$.
- $P[f]$ is said to be homogeneous if $\overline{d}(P) = \underline{d}(P)$.
- $P[f]$ is called a linear diferential polynomial generated by f if $\overline{d}(P) = 1$. Otherwise $P[f]$ is called non-linear differential polynomial. We denote by $Q = max\{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = max\{n_{1j} + 2n_{2j} + ... + kn_{kj} : 1 \leq j \leq t\}$.

In this paper, we will study the problem of a meromorphic function sharing one small function with its differential polynomial and obtain the following result which is an improvement of the above Theorem F of Lipeu Liu[11].

Theorem 1 : Let f be a non-constant meromorphic function, and $P[f]$ be a non-constant differential polynomial generated by f of degree $\overline{d}(P)$. Let $a \equiv a(z)(\not\equiv 0, \infty)$ be a small meromorphic function, such that $T(r,a) = S(r,f)$, as $r \to \infty$. Suppose that f^n and $P[f]$ share a IM. and

$$4\overline{N}(r,f) + 2\overline{N}(r, \frac{1}{(f^n/a)'}) + 5\overline{N}(r, \frac{1}{P[f]}) < (\lambda + o(1))T(r, P[f]) \tag{1.1}$$

or f^n and $P[f]$ share a CM and

$$2\overline{N}(r,f) + \overline{N}(r, \frac{1}{(f^n/a)'}) + 2\overline{N}(r, \frac{1}{P[f]}) < (\lambda + o(1))T(r, P[f]), \tag{1.2}$$

for $r \in I$, where I is a set of infinite linear measure and λ satisfies $0 < \lambda < 1$, then $\frac{P[f]-a}{f^n-a} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

2. Some Lemmas

For the proof of our results, we need the following lemmas:

Lemma 2.1([2], [9]) : Let $f(z)$ be a non-constant meromorphic function, k be a positive integer, then

$$\begin{aligned} N_p(r, \frac{1}{f^{(k)}}) &\le N_{p+k}(r, \frac{1}{f}) + k\overline{N}(r,f) + S(r,f) \\ &\le (p+k)\overline{N}(r, \frac{1}{f}) + k\overline{N}(r,f) + S(r,f) \end{aligned}$$

Lemma 2.2[4] : Let $f(z)$ be a non-constant meromorphic function, n be a positive integer. $Q(f) = a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f$, where a_i are meromorphic functions such that $T(r, a_i) = S(r,f)(i = 1, 2, ...n)$, $a_n \not\equiv 0$. Then
$T(r, Q(f)) = nT(r,f) + S(r,f)$.

Lemma 2.3[13] : Let f be a non-constant meromorphic function and let

$$R(f) = \frac{\Sigma_{k=0}^{n} a_k f^k}{\Sigma_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r,f) + S(r,f),$$

where $d = max\{n, m\}$.

Lemma 2.4[14] : Let f be a meromorphic function and $P[f]$ be a differential polynomial. Then $m(r, \frac{P[f]}{f^{\overline{d}(P)}}) \leq (\overline{d}(P) - \underline{d}(P))m(r, \frac{1}{f}) + S(r, f)$.

Lemma 2.5[14] : Let f be a meromorphic function and $P[f]$ be a differential polynomial. Then we have

$N(r, \frac{f^{\overline{d}(P)}}{P[f]}) \leq (\Gamma_p - \overline{d}(P))\overline{N}(r, f) + (\overline{d}(P) - \underline{d}(P))N_{(k+1}(r, \frac{1}{f}) + Q\overline{N}_{(k+1}(r, \frac{1}{f}) + \overline{d}(P)N_{k)}(r, \frac{1}{f}) + S(r, f)$.

3. Proof of Theorem 1

Let $F = \frac{f^n}{a}$, $G = \frac{P[f]}{a}$, then $F - 1 = \frac{f^n - a}{a}$, $G - 1 = \frac{P[f] - a}{a}$. Since f^n and $P[f]$ share a IM(CM), F and G share 1 IM(CM) except the zeros and poles of $a(z)$.

Define

$$H = (\frac{F''}{F'} - \frac{2F'}{F - 1}) - (\frac{G''}{G'} - \frac{2G'}{G - 1}), \tag{3.1}$$

we have the following two cases to investigate

Case 1 : $H \equiv 0$. On integration we get from (3.1)

$$\frac{1}{F - 1} \equiv C\frac{1}{G - 1} + D, \tag{3.2}$$

where C and D are constants and $C \neq 0$. If there exists a pole z_0 of f with multiplicity p which is not zero or pole of a, then z_0 is a pole of G with multiplicity $pd + (\Gamma - d)$, a pole of F with multiplicity p. This contradicts with (3.2) as Q contains at least one derivative. Therefore, we have

$$\overline{N}(r, f) \leq \overline{N}(r, a) + \overline{N}(r, \frac{1}{a}) = S(r, f), \tag{3.3}$$

$$\overline{N}(r, F) = S(r, f), \overline{N}(r, G) = S(r, f).$$

From (3.2), we also get that F and G share the value 1 CM.

Next we prove $D = 0$.

Suppose $D \neq 0$, then we have

$$\frac{1}{F - 1} \equiv \frac{D(G - 1 + \frac{C}{D})}{G - 1}, \tag{3.4}$$

So,

$$\overline{N}(r, \frac{1}{D(G - 1 + \frac{C}{D})}) = \overline{N}(r, F) = S(r, f) \tag{3.5}$$

If $\frac{C}{D} \neq 1$, by the second fundamental theorem and (3.5), and $S(r,G) = S(r,f)$, we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-1+\frac{C}{D}}) + S(r,G) \\ &\leq \overline{N}(r,\frac{1}{G}) + S(r,f) \leq N_2(r,\frac{1}{G}) + S(r,f) \leq T(r,G) + S(r,f) \end{aligned}$$

This gives that $N_2(r,\frac{1}{G}) = T(r,G) + S(r,f)$
So, we have $N_2(r,\frac{1}{P[f]}) = T(r,P[f]) + S(r,f)$
This contradicts with conditions (1.1),(1.2).
If $\frac{C}{D} = 1$, from (3.2) we know

$$\frac{1}{F-1} \equiv C\frac{G}{G-1}$$

then

$$(F-1-\frac{1}{C})G = -\frac{1}{C} \tag{3.6}$$

From (3.6) it follows that

$$N_{(k+1}(r,\frac{1}{f}) \leq N(r,\frac{1}{P[f]}) \leq N(r,\frac{1}{G}) = S(r,f) \tag{3.7}$$

Again from (3.6) we see that

$$\frac{1}{f^{\overline{d}(P)}(f^n-(1+\frac{1}{C})a)} \equiv \frac{-C}{a^2}.\frac{P[f]}{f^{\overline{d}(P)}}$$

Hence by the first fundamental theorem, (3.3),(3.7), Lemma 2.3, 2.4 and 2.5 we get that

$$\begin{aligned} (n+\overline{d}(P))T(r,f) &= T(r,f^{\overline{d}(P)}(f^n-(1+\frac{1}{C})a)) + S(r,f) \\ &= T(r,\frac{1}{f^{\overline{d}(P)}(f^n-(1+\frac{1}{C})a)}) + S(r,f) \\ &= T(r,\frac{P[f]}{f^{\overline{d}(P)}}) + S(r,f) \\ &\leq m(r,\frac{P[f]}{f^{\overline{d}(P)}}) + N(r,\frac{P[f]}{f^{\overline{d}(P)}}) + S(r,f) \\ &\leq (\overline{d}(P)-\underline{d}(P))[T(r,f) - N_{k)}(r,\frac{1}{f}) + N_{(k+1}(r,\frac{1}{f})] \\ &\quad +(\overline{d}(P)-\underline{d}(P))N_{(k+1}(r,\frac{1}{f}) + Q\overline{N}_{(k+1}(r,\frac{1}{f}) + \overline{d}(P)N_{k)}(r,\frac{1}{f}) + S(r,f) \\ &\leq (\overline{d}(P)-\underline{d}(P))T(r,f) + \underline{d}(P)N_{k)}(r,\frac{1}{f}) + S(r,f) \end{aligned} \tag{3.8}$$

From (3.8) it follows that $nT(r,f) \leq S(r,f)$, which is absurd.
Hence $D=0$ and so $\frac{G-1}{F-1} = C$ or $\frac{P[f]-a}{f^n-a} = C$
This proves the theorem.
Case 2 : $H \not\equiv 0$. From (3.1) it is to see that $m(r,H) = S(r,f)$.
Subcase 2.1 : Suppose that f^n and $P[f]$ share a IM, in this case, F and G share 1 IM except the zeros and poles of $a(z)$. We have

$$\overline{N}(r,F) = \overline{N}(r,f) + S(r,f) \quad , \quad \overline{N}(r,G) = \overline{N}(r,f) + S(r,f) \tag{3.9}$$

and by (3.1), we have

$$\begin{aligned} N(r,H) \leq \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_L(r,\frac{1}{F-1}) \\ + \overline{N}_L(r,\frac{1}{G-1}) + \overline{N}_0(r,\frac{1}{F'}) + \overline{N}_0(r,\frac{1}{G'}) + S(r,f). \end{aligned} \tag{3.10}$$

where $N_0(r,\frac{1}{F'})$ denotes the counting function of the zeros of F', which are not the zeros of F and $F-1$, and $\overline{N}_0(r,\frac{1}{F'})$ denotes its reduced form. In the same way, we can define $N_0(r,\frac{1}{G'})$ and $\overline{N}_0(r,\frac{1}{G'})$.
From the definitions of F and G, we get

$$N_{11}(r,\frac{1}{F-1}) = N_{11}(r,\frac{1}{G-1}) + S(r,f) \quad , \quad N_{(22)}(r,\frac{1}{F-1}) = N_{(22}(r,\frac{1}{G-1}) + S(r,f) \tag{3.11}$$

$$\begin{aligned} \overline{N}_L(r,\frac{1}{F-1}) &\leq N(r,\frac{1}{F-1}) - \overline{N}(r,\frac{1}{F-1}) \leq N(r,\frac{F}{F'}) \\ &\leq \overline{N}(r,\frac{F'}{F}) + S(r,f) \leq \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) + S(r,f) \end{aligned} \tag{3.12}$$

$$\begin{aligned} \overline{N}(r,\frac{1}{F}) &= \overline{N}(r,\frac{1}{G-1}) + S(r,f) \\ &\leq N_{11}(r,\frac{1}{F-1}) + \overline{N}_{(22}(r,\frac{1}{F-1}) \\ &\quad + \overline{N}_L(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{G-1}) + S(r,f) \end{aligned} \tag{3.13}$$

Let z_0 be a common simple zero of $F-1$ and $G-1$, but $a(z_0) \neq 0, \infty$.
But calculation, we know that z_0 is a zero of H, so

$$N_{11}(r,\frac{1}{F-1}) \leq N(r,\frac{1}{H}) + S(r,f) \leq T(r,H) + S(r,f) \leq N(r,H) + S(r,f) \tag{3.14}$$

From (3.10),(3.11),(3.12),(3.13) and (3.14), we have

$$\begin{aligned}\overline{N}(r,\frac{1}{G-1}) &\leq \overline{N}(r,F)+2\overline{N}_L(r,\frac{1}{F-1})+2\overline{N}_L(r,\frac{1}{G-1})+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})\\ &\quad +\overline{N}_{(22}(r,\frac{1}{F-1})+\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+S(r,f)\\ &\leq \overline{N}(r,F)+2\overline{N}(r,\frac{1}{F'})+2\overline{N}_L(r,\frac{1}{G-1})\\ &\quad +\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_0(r,\frac{1}{G'})+S(r,f) \qquad (3.15)\end{aligned}$$

By the second fundamental theorem, we have

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{G-1})-\overline{N}_0(r,\frac{1}{G'})+S(r,G)\\ &\leq 2\overline{N}(r,G)+2\overline{N}(r,\frac{1}{F'})+\overline{N}(r,\frac{1}{G})+\overline{N}_{(2}(r,\frac{1}{G})+2\overline{N}_L(r,\frac{1}{G-1})+S(r,G)\\ &\leq 2\overline{N}(r,G)+2\overline{N}(r,\frac{1}{F'})+\overline{N}(r,\frac{1}{G})+2\overline{N}(r,\frac{1}{G'})+S(r,f).\end{aligned}$$

By Lemma (2.1), and noting that $\overline{N}(r,\frac{1}{G'})=N_1(r,\frac{1}{G'})$, we get

$$T(r,P[f])\leq 4\overline{N}(r,f)+2\overline{N}(r,\frac{1}{(f^n/a)'})+5\overline{N}(r,\frac{1}{P[f]})+S(r,f).$$

which contradicts (1.1).

Subcase 2.2 : Suppose that f^n and $P[f]$ share a CM, in the case, F and G share 1 CM except the zeros and poles of $a(z)$. From (3.1), we have

$$N(r,H)\leq \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+S(r,f). \qquad (3.16)$$

Let z_0 be a common simple zero of $F-1$ and $G-1$, but $a(z_0)\neq 0,\infty$.
By calculation, we know that z_0 is a zero of H, so

$$\begin{aligned}N_{1)}(r,\tfrac{1}{F-1}) &\leq N(r,\tfrac{1}{H})+S(r,f)\\ &\leq N(r,H)+S(r,f)\end{aligned} \qquad (3.17)$$

Noting that
$N_{1)}(r,\frac{1}{F-1})=N_{1)}(r,\frac{1}{G-1})+S(r,f)$,
so

$$\begin{aligned}\overline{N}(r,\frac{1}{G-1}) &= N_{1)}(r,\frac{1}{F-1})+N_{(2}(r,\frac{1}{F-1})\\ &\leq \overline{N}(r,F)+N_{(2}(r,\frac{1}{F})+N_{(2}(r,\frac{1}{G})+N_{(2}(r,\frac{1}{F-1})\\ &\quad +N_0(r,\frac{1}{F'})+N_0(r,\frac{1}{G'})+S(r,f)\end{aligned}$$

By the second fundamental theorem, we can get

$$\begin{aligned} T(r,G) &\le \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-1}) - \overline{N}_0(r,\frac{1}{G'}) + S(r,G) \\ &\le 2\overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + N_{(2}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{F'}) + S(r,f) \\ &= 2\overline{N}(r,G) + N_{(2}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{F'}) + S(r,G) \end{aligned}$$

Similarly we have

$$T(r,P[f]) \le 2\overline{N}(r,f) + \overline{N}(r,\frac{1}{(f^n/a)'}) + 2\overline{N}(r,\frac{1}{P[f]}) + S(r,f)$$

This contradicts with (1.2). Theorem 1 thus completely proved.

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