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FIXED POINT AND COMMON FIXED POINT THEOREM ON BANACH SPACE USING ITERATIVE SCHEMES

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Abstract

The aim of this manuscript is to establish fixed point and common fixed point theorems satisfying contractive conditions of Banach spaces. The results proved there is the extension of some wellknown results in the existing literature.

1. Introduction

In 1976, Rhoades [15] introduced the convergence result of Zamfirescu operators using Mann and Ishikawa iterative schemes. Berinde [3] established the class of operators that is more elaborate than the class Zamfirescu operators. It introduced the convergence results of Ishikawa iteration process from this class of operators. After strong convergence of two-step iterative processes, in 2006, Rafiq [14] studied the convergence of quasi

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-contractive mappings by a three-step iterative scheme. Olatinwo [12] introduces convergence results of the class of generalized Zamfirescu operators under the Jungck-Ishikawa and the Jungck-Mann iteration scheme and Olatinwo [11] studied the convergence for generalized Zamfirescu operators by Jungck-Noor iterative scheme. Bosede [5] introduced strong convergence results of contractive-like mappings with the Jungck-Ishikawa and the Jungck-Mann iterative schemes.After many researchers has studied this concept in various ways. Many researchers studying strong convergence the following are Rhoades [15], Berinde [3,4], Olatinwo [11,12] Osilike and Udomene [13], Bosede [5] and Bele et al., [2]. The final concept of this task will study the intensity of iterative methods.

2. Preliminaries and Definitions

Firstly, useful definitions, theorems, and lemmas in our results.

In 1953, W. R. Mann [9] introduced the following iterative scheme and $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n$$

where, $\{\alpha_n\}, n \in N$ is the sequence of positive numbers in $[0, 1]$.

In 1974, Ishikawa [7] introduced the following iterative scheme and $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n$$

where, $\{\alpha_n\}, \}\beta_n\}, n \in N$ are the sequence of positive numbers in $[0, 1]$.

In 2000, Noor [10] introduced the following iterative scheme and $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n$$

$$y_n = (1 - \beta_n)X_n + \beta_n T z_n$$

$$z_n = (1 - \gamma_n)x_n + \gamma_n T x_n \tag{1}$$

where, $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}, n \in N$ are the sequence of positive numbers in $[0, 1]$.

We have to introduce the following new iterative scheme and $\{x_n\}$ defined by

$$X_{n+1} = (1 - \alpha_n)X_n + \alpha_n \ T_1 y_n$$

$$y_n = (1 - \beta_n)X_n + \beta_n T_2 z_n$$

$$z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n \tag{2}$$

where, $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}, n \in N$ are the sequence of positive numbers in [0,1].

Theorem 1.1 [16]: Let K be a non-empty closed convex subset of a metric space B and $T : K \to K$ be a mapping on K. Then the mapping T is called Zamfirescu operator if and only if there exists real numbers a, b, c such that

1. $d(Tx, Ty) \leq ad(x, y)$
2. $d(Tx, Ty) \leq b\{d(x, Tx) + d(y, Ty)\}$
3. $d(Tx, Ty) \leq c\{d(x, Ty) + d(y, Tx)\}$.

Then T has a unique fixed point q and the Picard iterative scheme $\{x_n\}$ defined by

$$x_{n+1} = Tx_n$$

converges to q for any arbitrary but fixed $x_0 \in K$.

In 2005, Berinde [3] discussed a new class of operators on metric space, Banach space and it is given by

$$\|Tx - Ty\| = 2\delta\|x - Tx\| + L\|x - y\|, \quad \forall\ x, y \in K \ \text{ and } \ \delta, L \in [0, 1) \tag{3}$$

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}, \quad 0 \leq \delta < 1.$$

Definition : Let K be a non-empty subset of a Banach space B and let $T : K \to K$ be a self-mapping of K and let $F = \{q \in K : Tq = q\}$ is the set of fixed points of T.

The contractive condition (3) was used by Olatinwo [12] to show that strong convergence results for Jungck-Ishikawa iteration process.

There exists a real number $\delta \in [0, 1)$ and a monotonic increasing function $\phi : R^+ \to R^+$ such that $\phi(0) = 0$ and $\forall\ \ x, y \in K$, we have

$$\|Tx - Ty\| \leq \phi(\|Sx - Tx\|) + \delta\|Sx - Sy\|. \tag{4}$$

We take $S = I$ in (4), the contractive mapping as follows.

There exists a real number $\delta \in [0, 1)$ and a monotonic increasing function $\phi : R^+ \to R^+$ such that $\phi(0) = 0$ and $\forall\ \ x, y \in K$, we have

$$\|Tx - Ty\| \leq \phi(\|x - Tx\|) + \delta\|x - y\| \tag{5}$$

Lemma: 1.1 [15] : Let p be a real number such that $0 \leq p < 1$ and $\{b_n\}$ be sequences of non negative real numbers such that $\lim_{n\to\infty} b_n = 0$.

3. Main Results

We now prove our main theorem as follows.

Theorem 3.2.1 : Let K be a non-empty closed convex subset of a Banach space B and $T : K \to K$ be a mapping satisfying (5) and $F(T) \neq \phi$. Let $\{x_n\}_{n=0}^{\infty}$ be defined by iteration scheme (1). If $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences of positive numbers in $[0, 1]$ such that $\sum_{n=0}^{\infty} a_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to fixed points of T.

Proof: Let $q \in F(T)$ then $\{x_n\}_{n=0}^{\infty}$ we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1 - a_n)x_n + a_n Ty_n - q\| \\ &\leq (1 - a_n)\|x_n - q\| + a_n\|Ty_n - q\| \\ &\leq (1 - a_n)\|x_n - q\| + a_n\phi\|Tq - q\| + a_n\delta a_n\|y_n - q\| \\ &= (1 - a_n)\|x_n - q\| + a_n\delta\|y_n - q\|. \qquad (6) \end{aligned}$$

Now

$$\begin{aligned} \|z - q\| &= \|(1 - c_n)x_n + c_n Tx_n - q\| \\ &\leq (1 - c_n)\|x_n - q\| + c_n\|Tx_n - q\| \\ &\leq (1 - c_n)\|x_n - q\| + c_n\delta\|x_n - q\| + c_n\phi(\|Tq - q\|) \\ &= (1 - c_n)\|x_n - q\| + c_n\delta\|x_n - q\| \\ &= (1 - (1 - \delta)c_n)\|x_n - q\| \qquad (7) \end{aligned}$$

and

$$\begin{aligned} \|y_n - q\| &= \|(1 - b_n)x_n + b_n Tx_n - q\| \\ &\leq (1 - b_n)\|x_n - q\| + b_n\|Tz_n - q\| \\ &\leq (1 - b_n)\|x_n - q\| + b_n\delta\|z_n - q\| + b_n\phi(\|Tq - q\|) \\ &= (1 - b_n)\|x_n - q\| + b_n\delta\|z_n - q\| \qquad (8) \end{aligned}$$

From equation (7) and (8)

$$\begin{aligned} \|y_n - q\| &\leq (1 - b_n\|x_n - q\| + b_n\delta(1 - (1 - \delta)c_n)\|x_n - q\| \\ &= [(1 - b_n) + b_n\delta(1 - (1 - \delta)c_n)]\|x_n - q\|. \qquad (9) \end{aligned}$$

Therefore using (9) and (6) we obtain

$$\begin{aligned}\|x_{n+1}-q\| &\leq (1-a_n)\|x_n-q\|+a_n\delta\|y_n-q\| \\ &\leq (1-a_n)\|x_n-q\|+a_n\delta[(1-b_n)+b_n\delta(1-(1-\delta)c_n)]\|x_n-q\| \\ &= [1-(1-\delta)a_n-(1-\delta)b_na_n\delta-(1-\delta)c_nb_na_n\delta^2]\|x_n-q\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|x_{n+1}-q\| &\leq [1-(1-\delta)a_n]\|x_n-q\| \\ &\leq \prod_{i=0}^{n}[1-(1-\delta)a_i]\|x_0-q\| \\ &\leq \|x_0-q\|\exp\left(\sum_{i=0}^{n}-(1-\delta)a_i\right). \qquad (10)\end{aligned}$$

Since $0<\delta<1$, $\alpha_i\in[0,1]$ and $\sum\limits_{n=0}^{\infty}a_n=\infty$, so $\exp\left(\sum\limits_{i=0}^{n}-|(1-\delta)a_i\right)\to 0$ as $n\to\infty$. Hence, it follows from (10) and Lemma 1.1 we have $\lim\limits_{n\to\infty}\|x_{n+1}-q\|=0$.

Thus, $\{x_n\}_{n=0}^{\infty}$ converges strongly to q of the fixed point of T.

Theorem 3.2.2 : Let K be a non-empty closed conved subset of a Banach space B and $T_1,T_2,T_3:K\to K$ be a mapping satisfying (5) and $\bigcap\limits_{i=1}^{3}F|(T_i)\neq\phi$. Let $\{x_n\}_{n=0}^{\infty}$ be defined by iteration scheme (2). If $\{a_n\},\{b_n\}$ and $\{c_n\}$ are sequences of positive numbers in $[0,1]$ such that $\sum\limits_{n=0}^{\infty}a_n=\infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T_1, T_2 and T_3.

Proof : Let $q\in F(T)$ then $\{x_n\}_{n=0}^{\infty}$ we have

$$\begin{aligned}\|x_{n+1}-q\| &= \|(1-a_n)x_n+a_nT_1y_n-q\| \\ &\leq (1-a_n)\|x_n-q\|+a_n\|T_1y_n-q\| \\ &\leq (1-a_n)\|x_n-q\|+a_n\delta\|y_n-q\|+a_n\phi(\|T_1q-q\|) \\ &= (1-a_n)\|x_n-q\|+a_n\delta\|y_n-q\|. \qquad (11)\end{aligned}$$

Now,

$$\begin{aligned}
\|z_n - q\| &= \|(1-c_n)x_n + c_nT_3x_n - q\| \\
&\le (1-c_n)\|x_n - q\| + c_n\|T_3x_n - q\| \\
&\le (1-c_n)\|x_n - q\| + c_n\delta\|x_n - q\| + c_n\phi(\|T_3q - q\|) \\
&= (1-c_n)\|x_n - q\| + c_n\delta\|x_n - q\| \\
&= (1-(1-\delta)c_n)\|x_n - q\| \qquad (12)
\end{aligned}$$

$$\begin{aligned}
\|y_n - q\| &= \|(1-b_n)x_n + b_nT_2x_n - q\| \\
&\le (1-b_n)\|x_n - q\| + b_n\|Tz_n - q\| \\
&\le (1-b_n)\|x_n - q\| + b_n\delta\|z_n - q\| + b_n\phi(\|Tq - q\|) \\
&= (1-b_n)\|x_n - q\| + b_n\delta\|z_n - q\|. \qquad (13)
\end{aligned}$$

From equation (12) and (13)

$$\begin{aligned}
\|y_n - q\| &\le (1-b_n)\|x_n - q\| + b_n\delta(1-(1-\delta)c_n)\|x_n - q\| \\
&= [(1-b_n) + b_n\delta(1-(1-\delta)c_n)]\|x_n - q\|. \qquad (14)
\end{aligned}$$

Therefore using (14) and (11), we obtain

$$\begin{aligned}
\|x_{n+1} - q\| &\le (1-a_n)\|x_n - q\| + a_n\delta\|y_n - q\| \\
&\le (1-a_n)\|x_n - q\| + a_n\delta[(1-b_n) + b_n\delta(1-(1-\delta)c_n)]\|x_n - q\| \\
&\le [1-(1-\delta)a_n - (1-\delta)b_na_n\delta - (1-\delta)c_nb_na_n\delta^2]\|x_n - q\| \\
&\le [1-(1-\delta)a_n]\|x_n - q\| \\
&\le \prod_{i=0}^{n}[1-(1-\delta)a_i]\|x_0 - q\| \\
&\le \|x_0 - q\| \exp\left(\sum_{i=0}^{n} -(1-\delta)a_i\right). \qquad (15)
\end{aligned}$$

Since $0 \le \delta < 1$, $a_i \in [0,1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, so $\exp\left(\sum_{i=0}^{n} -(1-\delta)a_i\right) \to 0$ as $n \to \infty$.
Hence, it follows from (15) and Lemma 1.1 we have $\lim_{x\to\infty} \|x_{n+1} - q\| = 0$.
Therefore, $\{x_n\}_{n=0}^{\infty}$ converges strongly to q of the fixed point of T_1, T_2 and T_3.

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