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ON CERTAIN NEW COLOR PARTITION IDENTITIES DEDUCED FROM SOMO'S THETA FUNCTION OF LEVEL 15

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Abstract

M. Somos discovered around 6277 theta-function identities of different levels using computer and offered no proof for them and these identities highly resembles Ramanujan's recordings. The purpose of this paper is to prove ten of his identities of level 15 and to establish color partition identities from them.

1. Introduction

Throughout this paper, we assume $|q| < 1$. Let

$$f(-q) = \prod_{n=1}^{\infty}(1 - q^n).$$

For $q = e^{2\pi ir}$, $f(-q) = e^{\frac{-\pi ir}{12}}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind η-function for $Im(\tau) > 0$. For convenience we set $f(-q^k) = f_k$.

Ramanujan recorded several identities which involve $f(-q)$, $f(-q^n)$, $f(-q^m)$ and $f(-q^{mn})$ called level mn in his second notebook [3] and Lost Notebook [4]. For example

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$$f_1^4f_2^4f_5^2f_{10}^2 + 5f_1^2f_2^2f_5^4f_{10}^4 = f_2^6f_5^6 + f_1^6f_{10}^6.$$

Michael Somos recently used a computer to discover several new elegant theta-function identities in the spirit of Ramanujan and offered no proof for them. Somos has a large list of η-product identities and he runs PARI/GP scripts to look at each identity in $P-Q$ forms. Recently B. Yuttanan [6] has proved certain Somos theta-function identities of different levels by employing Ramanujan's modular equations and K. R. Vasuki and R. G. Veeresha [5] proved η-function identities of level 14 discovered by Somos.

The purpose of this paper, is to prove all the theta function identities of level 15 conjectured by Somos. Further we establish certain interesting color partition identities from them.

Now we shall recall some known theta function identities of Ramanujan which will be used in the sequel to prove Somos identities. For

$$X = \frac{f_1}{q^{\frac{1}{12}}f_3}, Y = \frac{f)5}{q^{\frac{5}{12}}f_{15}} \quad \text{and} \quad \frac{f_3}{q^{\frac{1}{2}}f_{15}}. \tag{1.1}$$

we have **Theorem 1.1** [[1], p. 221] [3, p. 325]

$$(XY)^2 + 5 + \frac{9}{(XY)^2} = \left(\frac{Y}{X}\right)^3 - \left(\frac{X}{Y}\right)^3. \tag{1.2}$$

Theorem 1.2 : [1, p. 223] [3, p. 323]

$$(PQ)^3 + \frac{125}{(PQ)^3} = \left(\frac{Q}{P}\right)^6 - 9\left(\frac{Q}{P}\right)^3 - 9\left(\frac{P}{Q}\right)^3 - \left(\frac{P}{Q}\right)^6. \tag{1.3}$$

Theorem 1.3 [1, p.226].

$$(PQ)^3 - \frac{125}{(PQ)^3} = (XY)^4 + (XY)^2 - \frac{9}{(XY)^2} - \frac{81}{(XY)^4}. \tag{1.4}$$

Somos also rediscovered the above three identities, a proof of these identities can be found in [1, pp. 221-230].

In Section 2, we prove Somos identities of level 15. From these identities, in Section 3 we deduce certain interesting color partition identities.

2. Somos Identities of Level 15

In this section we state and prove Somos theta function identities of level 15.

Theorem 2.1 : We have

$$f_1^3f_3^6f_5^2+10q^2f_3^3f_5^5f_{15}^3-f_1^8f_3f_5f_{15}-5qf_1^2f_3f_5^7f_{15}-5q^3f_1^3f_5^2f_{15}^6-90q^4f_1f_3^2f_{15}^8=0. \quad (2.1)$$

Proof : Dividing (2:1) by $f_1^8f_3f_5f_{15}$ and using (1.1), we obtain

$$(PQ)^3=\frac{5[Y^5(2-(XY)^2)-X(X^2Y^2+18)]}{Y^4(X^5-Y)}. \quad (2.2)$$

Using the fact that $\frac{P}{Q}=\frac{X}{Y}$ in (1.3), we have

$$(PQ)^3+\frac{125}{(PQ)^3}=\left(\frac{Y}{X}\right)^6-9\left(\frac{Y}{X}\right)^3-\left(\frac{X}{Y}\right)^6.$$

Employing (2.2) in the above twice and factorizing using maple, we find that $(X^5Y^5+X^6+5X^3Y^3-Y^6+9XY)\,(X^{14}Y^2-16X^{11}Y^5+4X^5Y^{11}-X^2Y^{14}+18X^{12}+62X^9Y^3+20X^6Y^6-18X^3Y^9+2Y^{12}-180X^7Y)=0$.

From (1.3) the above identity holds. This implies (2.1) holds.

Theorem 2.2 : We have

$$qf_1^6f_3^2f_{15}^3+2f_1^8f_5^2f_{15}+225q^4f_1f_3f_5f_{15}^8+9qf_1f_3^7f_5f_{15}^2-2f_1^3f_3^5f_5^3-25q^2f_3^2f_5^6f_{15}^3=10. \quad (2.3)$$

$$f_1^2f_3^6f_5^3+18qf_3^8f_5f_{15}^2+2qf_1^5f_3^3f_{15}^3-f_1^7f_3f_5^2f_{15}-25qf_1f_3f_5^8f_{15}-25q^3f_1^2f_5^3f_{15}^6=0. \quad (2.4)$$

$$f_1^6f_3^3f_{15}^2+f_3^{10}f_5^3+100q^4f_1^3f_3f_{15}^9-2f_1^5f_3^5f_5^2f_{15}-25q^2f_1^2f_3^2f_5^5f_{15}^4=0. \quad (2.6)$$

$$f_1^{10}f_5f_{15}^2+18f_1^5f_3^5f_5f_{15}^2+81f_3^{10}f_5^2f_{15}-100-f_1f_3^3f_5^9-225q^2f_1^2f_3^2f_5^4f_{15}^5=0. \quad (2.7)$$

$$f_1^5f_3^4f_5^2f_{15}^2+4f_3^9f_5^3f_{15}-10q^2f_1^2f_3f_5^5f_{15}^5-5q^4f_1^3f_{15}^{10}-5f_1f_3^2f_5^{10}=0. \quad (2.8)$$

$$4f_1^9f_5f_{15}^3+405q^4f_1^2f_3f_{10}^{15}+5f_3^3f_5^{10}-9f_1^4f_3^5f_5^2f_{15}^2-90q^2f_1f_3^2f_5^5f_{15}^5=0. \quad (2.9)$$

$$qf_1^8f_3^2f_{15}^4+f_1^{10}f_5^2f_{15}^2+125q^4f_1^3f_3f_5f_{15}^9+9qf_1^3f_3^7f_5f_{15}^3-f_3^{10}f_5^4=0. \quad (2.10)$$

$$f_1^9f_3f_5f_{15}^3+81q^4f_1^2f_3^2f_{15}^{10}+9qf_1^3f_3f_5^7f_{15}^3+9q^3f_1^4f_5^2f_{15}^8-f_3^4f_5^{10}=0. \quad (2.11)$$

We are omitting the proof of identities from (2.3) to (2.11) as they are same as the proof of (2.1).

3. Color Partition

The Somos's identities, proved in Section 2, have interesting applications to color partition. Sen-Shan Huang introduced color partition in [2]. A positive integer n has k

colors if there are k copies of n and all of them are viewed as distinct objects. Partition of a positive integer into parts with colors are called "colored partitions".
For example, if 1 is allowed to have 2 colors, then all the (colored) partitions of 2 are 2; $1_r + 1_r; 1_g + 1_g$ and $1_r + 1_g$ where we use the indices r (red) and g (green) to distinguish two copies of 1.
The generating function for the number of partitions of n, where all the parts are congruent to u (mod v) and have k color is

$$\frac{1}{(q^u; q^v)_\infty^k},$$

where

$$(a; q)_\infty := \prod_{k=0}^{\infty}(1 - aq^n).$$

Definition 3.1 : Let $P(n, k, l, m)$ denote the number of partition of n into parts not congruent to 0 (mod 15) , with parts congruent to 0 (mod 3) having k colors and parts congruent to 0 (mod 5) having l colors and parts not congruent to 0 (mod 3) or 0 (mod 5) having m colors.
Definition 3.2 : We define

$$(a_1, a_2, a_3, \cdots, a_n; q)_\infty := \prod_{k=1}^{n}(a_k; q0_\infty$$

and

$$(q^{r_1\pm}, q^{r_2\pm}, q^{r_3\pm}, \cdots, q^{r_n\pm}; q^s)_\infty := (q^{r_1}, q^{r_2}, \cdots, q^{r_n}, q^{s-r_1}, q^{s-r_2}, \cdots, q^{s-r_n}; q^s)_\infty$$

where $r_i < s$ with $1 \le i \le n$.
For example,

$$(q^{1\pm}, q^{2\pm}, q^{3\pm}, q^{4\pm}, q^{5\pm}, q^{6\pm}, q^{7\pm}; q^{15})_\infty = (q^1, q^2, q^3, q^4, \cdots, q^{14}; q^{15})_\infty.$$

Theorem 3.3 : We have, for $n \ge 4$

$$\begin{aligned} P(n, 2, 6, 8) + 10P(n-2, 8, 6, 11) &= P(n, 2, 2, 3) + 5P(n-1, 8, 2, 9) \\ &\quad +5P(n-3, 8, 6, 8) + 90P(n-4, 8, 11, 11), \end{aligned}$$

where $P(0, k, l, m) = 1$.

Proof : Dividing (2.1) by f_1^{11}, we find that

$$\left(\frac{f_3}{f_1}\right)^6\left(\frac{f_5}{f_1}\right)^2+10q^2\left(\frac{f_3}{f_1}\right)^3\left(\frac{f_5}{f_1}\right)^5\left(\frac{f_{15}}{f_1}\right)^3-\left(\frac{f_3}{f_1}\right)\left(\frac{|f_5}{f_1}\right)\left(\frac{f_{15}}{f_1}\right)$$
$$-5q\left(\frac{f_3}{f_1}\right)\left(\frac{f_5}{f_1}\right)^7\left(\frac{f_{15}}{f_1}\right)-5q^3\left(\frac{f_5}{f_1}\right)^2\left(\frac{f_{15}}{f_1}\right)^6-90q^4\left(\frac{f_3}{f_1}\right)^2\left(\frac{f_{15}}{f_1}\right)^8=0.$$

This implies

$$\frac{1}{(q_8^{1\pm},q_8^{2\pm},q_2^{3\pm},q_8^{4\pm},q_6^{5\pm},q_2^{6\pm},q_8^{7\pm};q^{15})_\infty}+\frac{10q^2}{(q_{11}^{1\pm},q_{11}^{2\pm},q_8^{3\pm},q_{11}^{4\pm},q_6^{5\pm},q_8^{6\pm},q_{11}^{7\pm};q^{15})_\infty}$$
$$-\frac{1}{(q_3^{1\pm},q_3^{2\pm},q_2^{3\pm},q_3^{4\pm},q_2^{5\pm},q_2^{6\pm},q_3^{7\pm};q^{15})_\infty}-\frac{5q}{(q_9^{1\pm},q_9^{2\pm},q_8^{3\pm},q_9^{4\pm},q_2^{5\pm},q_8^{6\pm},q_9^{7\pm};q^{15})_\infty}$$
$$-\frac{5q^3}{(q_8^{1\pm},q_8^{2\pm 2},q_8^{3\pm},q_8^{4\pm},q_6^{5\pm},q_8^{6\pm},q_8^{7\pm};q^{15})_\infty}-\frac{90q^4}{(q_{10}^{1\pm},q10^{2\pm},q_8^{3\pm},q_{10}^{4\pm},q_{10}^{5\pm},q_8^{6\pm},q_{10}^{7\pm};q^{15})_\infty}=0.$$

Employing the definition of $P(n,k,l,m)$ in the above, we obtain

$$\sum_{n=0}^{\infty}P(n,2,6,8)q^n+10\sum_{n=0}^{\infty}P(n-2,8,6,11)q^n-\sum_{n=0}^{\infty}P(n,2,2,3)q^n$$
$$-5\sum_{n=0}^{\infty}P(n-1,8,2,9)q^n-5\sum_{n=0}^{\infty}P(n-3,8,6,8)q^n-90\sum_{n=0}^{\infty}P(n-4,8,11,11)q^n=0.$$

On comparing the coffcients of q^n, we obtain the required result. □

Similarly as above we can deduce from (2.3) to (2.11) and (1.2),(1.3), (1.4) the following color partition identities respectively:

Theorem 3.4 : We have for $n \geq 4$

$$2P(n,3,5,8)+25P(n-2,9,5,11)=P(n-1,3,5,5)+2P(n,3,1,3)$$
$$+225P(n-4,9,9,10)+P(n-1,3,9,10).$$

We have for $n \geq 3$

$$P(n,3,6,9)+18P(n-1,3,10,11)+2P(n-1,3,6,6)=25P(n-1,9,2,10)$$
$$+P(n,3,2,4)+25P(n3,6,6,9).$$

We have for $n \geq 3$

$$P(n,2,5,5)+45P(n-3,8,9,10)+9P(n,2,9,10)=10P(n-2,8,5,8)$$
$$+10P(n,8,1,9)+5P(n-1,8,5,11).$$

We have for $n \geq 4$

$$P(n,3,1,3) + P(n,3,10,13) + 100P(n,4,9,10,12) = 2P(n,3,6,8) + 25P(n-2,9,6,11).$$

We have for $n \geq 2$

$$P(n,3,3,3) + 18P(n,3,7,8) + 81P(n,3,11,13) = 100P(n,9,3,12) + 225P(n-2,8,6,10).$$

We have for $n \geq 4$

$$P(n,4,6,9) + 4P(n,4,10,13) = 10P(n-2,10,6,11) + 5P(n-4,10,10,10) + 5P(n,10,12,12).$$

We have for $n \geq 4$

$$4P(n,4,3,4) + 45P(n-4,10,11,11) + 5P(n,10,3,13) = 9P(n,4,7,9) + 90P(n-2,10,7,12).$$

We have for $n \geq 4$

$$P(n,10,4,14) = P(n,4,4,5) + 81P(n-4,10,12,12) + 9P(n-1,10,4,11) + 9P(n-3,10,8,10).$$

We have for $n \geq 2$

$$P(n,6,12,12) = P(n-2,6,6,6) + P(n,6,2,7) + 5P(n-1,6,6,9) + 9P(n-2,6,10,11).$$

We have for $n \geq 4$

$$P(n,12,12,24) = P(n-4,12,12,12) + P(n,6,12,12) + 125P(n-4,18,12,21) + 9P(n-1,12,12,21) + 9P(n-3,12,12,15).$$

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