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COMMON FIXED POINT THEOREMS IN b-METRIC SPACE WITH THE HELP OF RATIONAL TYPE CONTRACTION MAPPING

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Abstract

In our paper, Motivated by the extended b-metric spaces concept, we prove main results regarding common fixed point theorems in generalized b-metric space. Which hold contraction mappings, M. Seddik and H. Taieb worked and showed that the results in generalized b-metrie space hold Cauchy sequence. Out results may further generalized some extended results.

1. Introduction

The concept of distance between two abstract objects has received importance not only

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for mathematical analysis and Generalization to an also for its related fields, b-metric space is one of many usual metric, which was introduced by Bakhtin in 1989 his work and then extensively used by Czerwik in 1993, Thereafter, lot of improvements have been done in finding fixed points for single and multi-valued operator in that space.
There are a lot of extensions of the notions of metric and metric space. In this paper we concentrate on b-metric and generalized b-metric spaces.
Some problems particularly the problem of the convergence of measurable functions with respect to measure leads Czerwik [10] to the generalization of metric space and introduced the concept of b-metric space and proved Banach's contraction theorem in so called b-metric space. After Czerwik [6] many papers have been published containing fixed point results on b- metric spaces and using the results [1, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14]..

2 Definitions and Preliminaries

Definition 2.1 ([2], [11]) : Let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the conditions.

1. $d(x, x) = 0$;
2. $d(x, y) = d(y, x) = 0$ implies that $x = y$;
3. $d(x, y) = d(y, x)$;
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d satisfies the conditions from 1 to 4 then it is called metric on X, if d satisfies conditions 2 to 4 then it is called dislocated metric (d-metric) on X and if d satisfies conditions 2 and 4 only then it is called dislocated quasi metric on X. In the view of above definition one can define the b-metric space [7] as following :
Example 2.1 [7] : Let $X = \{0, 1, 2\}$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0, d(1, 2) = d(2, 1) = d(0, 1) = d(1, 0) = 1, d(2, 0) = d(0, 2) = m \geq 2$ for $k = m2$ where $m \geq 2$ the function defined above is a b metric but not a metric for $m > 2$.
Definition 2.2 : Let X be a non-empty set, let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called b metric if $\forall \;\; x, y, z \in X$,

1. $d(x, x) = 0$;

2. $d(x,y) = d(y,x) = 0$ implies that $x = y$;

3. $d(x,y) = d(y,x)$;

4. $d(x,y) \leq k[d(x,z) + d(z,y)]$.

The pair (X,d) is called b-metric space. It is crystal clear from the above definition that b-metric is more generalization of metric. Evidently every metric is b-metric but the converse is not true as shown by the following example.

Example 2.2 : Let $0 < p < 1$ and define $L_p[a,b]$ by

$$L_p[a,b] = \left\{ x(t) | \int_a^b |x(t)|^p dt < \infty \right\}$$

where the mapping $d : L_p[a,b] \times L_p[a,b] \to R^+$ is define by

$$d(x,y) = \left(\int_a^b |x(t) - y(t)|^p \right)^{1/p}$$

for each $x = x(t), y = y(t) \in L_p[a,b]$, then $(L_p[a,b], d)$ is a b-metric space with coefficients $= 2^{\frac{1}{p}-1}$.

Lemma 2.1 : Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a mapping, consider that $\{P_n\}$ is a sequence in X induced by $P_{n+1} = fP_n$ we have

$$d(P_n, P_{n+1}) \leq \mu d(P_{n-1}, P_n) \tag{1}$$

for all $n \in N$, where $\mu \in (0,1)$ is a constant, then $\{P_n\}$ is a Cauchy sequence.

3. Main Results

Theorem 3.1 : Let (X,d) be a complete b-metric space with a coefficient $s \geq 1$ and $A, B : X \to X$ be self mappings on X, following conditions satisfying

$$\begin{aligned} d(Ap, Bp) \quad \leq \quad & a_1 d(p,q) + a_2 \frac{d(p,Ap)d(p,Bq) + d(q,Bq)d(q,Ap)}{d(p,Bq) + d(q,Ap)} \\ & + a_3 \frac{d(Ap,Bq)d(p,Ap) + d(p,Aq)d(Aq,Bq)}{d(Ap,Bq) + d(q,Ap)} \\ & + a_4 \frac{d(p,Ap)d(Bp,Bq) + d(Aq,Bp)d(Bp,q)}{d(Ap,Bq) + d(q,Ap)} \end{aligned} \tag{2}$$

For all p,q in X and $a_1, a_2, a_3, a_4 \geq 0$. Also we show that $d(p,Bq) + d(q,+Ap) \neq 0$, $d(Ap,Bq) + d(q,Ap) \neq 0, d(Ap,Bq) + d(q,Ap) \neq 0$, with $(a_1 + a_2 + a_3 + a_4) < 1$. Then A and B have a unique common fixed point in X.

Proof : Let we have any arbitrary point $p_0 \in X$, define sequence $\{p_n\}$ in X, s.t.

$$P_{2n+1} = AP_{2n}, P_{2n+2} = BP_{2n+1}, \quad \text{for all} \quad n \in N. \tag{3}$$

Consider, we have some $n \in N$, s.t. $P_n = P_{n+1}$, if $n = 2i$, then $P_{2i} = P_{2i+1}$. Now from the equation (2) and with help of $p = P_{2i}$ and $q = P_{2i+1}$, we get

$$\begin{aligned} d(P_{2i+1}, P_{2i+2}) \quad &= \quad d(A_{p2i}, B_{p2i+1}) \leq a_1 d(P_{2i} > P_{2i+1}) \\ &\quad +a_2 \frac{d(p_{2i}, Ap_{2i})d(p_{2i}, Bp_{2i+1}) + d(p_{2i+1}, Bp_{2i+1})d(p_{2i+1}, Ap_{2i})}{d(p_{2i}, Bp_{2i+1}) + d(p_{2i+1}, Ap_{2i})} \\ &\quad +a_3 \frac{d(Ap_{2i}, Bp_{2i+1})d(p_{2i}, A_{p_{2i}}) + d(p_{2i}, Ap_{2i+1})d(Ap_{2i+1}, Bp_{2i})}{d(Ap_{2i}, Bp_{2i+1}) + d(p_{2i+1}, Ap_{2i})} \\ &\quad +a_4 \frac{d(p_{2i}, Ap_{2i})d(Bp_{2i}, Bp_{2i+1}) + d(Ap_{2i+1}, Bp_{2i})d(Bp_{2i}, p_{2i+1})}{d(Ap_{2i}, Bp_{2i+1}) + d(p_{2i+1}, Ap_{2i})} \\ &= \quad a_1 d(p_{2i}, p_{2i+1}) \\ &\quad +a_2 \frac{d(p_{2i}, p_{2i+1})d(p_{2i}, p_{2i+2}) + d(p_{2i}, p_{2i+2})d(p_{2i+1}, p_{2i+1})}{d(p_{2i}, p_{2i+2}) + d(p_{2i+1}, p_{2i+2})} \\ &\quad +a_3 \frac{d(p_{2i+1}, p_{2i+1})d(p_{2i}, p_{2i+1}) + d(p_{2i}, p_{2i+2})d(p_{2i+2}, p_{2i+2})}{d(p_{2i+1}, p_{2i+2}) + d(p_{2i+1}, p_{2i+2})} \\ &\quad +a_4 \frac{d(p_{2i}, p_{2i+1})d(p_{2i+1}, p_{2i+2}) + d(p_{2i+2}, p_{2i+1})d(p_{2i+1}, p_{2i+1})}{d(p_{2i+1}, p_{2i+2}) + d(p_{2i+1}, p_{2i+2})} \\ &= 0. \end{aligned}$$

Then $d(p_{2i+1}, p_{2i+2}) = 0$, hence $p_{2i+1} = p_{2i+2}$. Thus we get $p_{2i} = p_{2i+1} = p_{2i+2}$, by equatiion (3), it means $p_{2i} = Ap_{2i} = Bp_{2i}$ s.t. p_{2i} is a common fixed point of A and B.

If $n = p_{2i+1}$, then we can apply same arguments as $p_{2i} + p_{2i+1}$, which is show, p_{2i+1} is a common fixed point of A and B.

Now, we consider that $p_n \neq p_{n+1}$, for all $n \in N$.

Result 1 : We have

$$d(p_n + p_{n+1}) \leq (a_1 + a_2 + a_3 + a_4) d(p_{n-1}, p_n) \quad \text{for all} \quad n \in N, \tag{4}$$

so that we can show some terms in given conditiion.

Condition 1 : Let $n = 2i + 1$, from equation (2) with $p = p_{2i}$ and $q = p_{2i+1}$, we get

$$\begin{aligned} d(p_{2i+1}, p_{2i+2}) &= d(Ap_{2i}, Bp_{2i+1}) \leq a_1 d(p_{2i}, p_{2i+1}) \\ &\quad + a_2 \frac{d(p_{2i}, Ap_{2i})d(p_{2i}, Bp_{2i+1}) + d(p_{2i+1}, Bp_{2i+1})d(p_{2i+1}, Ap_{2i})}{d(p_{2i}, Bp_{2i+1}) + d(p_{2i+1}, Ap_{2i})} \\ &\quad + a_3 \frac{d(Ap_{2i}, Bp_{2i+1})d(p_{2i}, Ap_{2i}) + d(p_{2i1}, Ap_{2i+1})d(Ap_{2i+1}, Bp_{2i+1})}{d(Ap_{2i}, Bp_{2i+1}) + d(p_{2i+1}, Ap_{2i})} \\ &\quad + a_4 \frac{d(p_{2i}, Ap_{2i})d(Bp_{2i}, Bp_{2i+1}) + d(Ap_{2i+1}, Bp_{2i})d(Bp_{2i}, p_{2i+1})}{Ad(p_{2i}, Bp_{2i+1}) + d(p_{2i+1}, Ap_{2i})} \\ &= a_1 d(p_{2i}, p_{2i+1}) \\ &\quad + a_2 \frac{d(p_{2i}, p_{2i+1})d(p_{2i}, p_{2i+2}) + d(p_{2i+1}, p_{2i+2})d(p_{2i+1}, p_{2i+1})}{d(p_{2i}, p_{2i+2}) + d(p_{2i+1}, p_{2i+1})} \\ &\quad + a_3 \frac{d(p_{2i+1}, p_{2i+2})d(p_{2i}, p_{2i+1}) + d(p_{2i}, p_{2i+2})d(p_{2i+2}, p_{2i+2})}{d(p_{2i+1}, p_{2i+2}) + d(p_{2i+1}, p_{2i+2})} \\ &\quad + a_4 \frac{d(p_{2i}, p_{2i+1})d(p_{2i+1}, p_{2i+2}) + d(p_{2i+2}, p_{2i+1})d(p_{2i+1}, p_{2i+1})}{d(p_{2i+1}, p_{2i+2}) + d(p_{2i}, p_{2i+1})} \\ &= (a_1 + a_2 + a_3 + a_4) d(p_{2i}, P_{2i+1}). \end{aligned}$$

Hence,

$$d(p_n, p_{n+1}) \leq (a_1 + a_2 + a_3 + a_4) d(p_{n-1}, p_n), \quad \text{for all} \quad n \in N. \tag{5}$$

Condition 2 : Let $n = 2i$ in condition (1), we use some argument, which can prove that in equation (4), dfor $n = 2_i$ s.t.

$$d(p_n, p_{n+1}) \leq (a_1 + a_2 + a_3 + a_4) d(p_{n-1}, p_n), \quad n = 2_i + 1. \tag{6}$$

Now, by equation (5) and (6), we get

$$d(p_n, p_{n+1}) \leq (a_1 + a_2 + a_3 + a_4) d(p_{n+1}, p_n), \quad n = N, \tag{7}$$

which is also show in equation (4).

Since $(a_1 + a_2 + a_3 + a_4) \leq 1$, by Lemma 2.1, we can say that $\{p_n\}$ is a Cauchy sequence in (X, d), so that (X, d) is a complete b-metric space $\{p_n\}$ converges to some $u \in X$ as $n \to \infty$.

Result 2 : Now, we can prove that $Aw = Bw = W,$.

By help of the triangular inequality and equatiion (2), we get

$$\begin{aligned}
d(w,Aw) &\leq s[d(w,p_{2n+2})+d(p_{2n+2},AW)]\\
&= sd(w,P_{2n+2})+sd(Aw,Bp_{2n+1})\\
&\leq sd(w,p_{2n+2})+sa_1\ d(w,p_{2n+1})\\
&\quad +sa_2\frac{d(w,Aw)d(w,Bp_{2i+1})+d(p_{2n+1},Bp_{2n+1})d(p_{2n+1},Aw)}{d(w,Bp_{2i+1})+d(p_{2i+1},Aw)}\\
&\quad +sa_3\frac{d(Aw,Bp_{2n+1})d(w,Aw)+d(w,Ap_{2n+1})d(Ap_{2n+1},Bp_{2n+1})}{d(Aw,Bp_{2n+1})+d(p_{2n+1},Aw)}\\
&\quad +sa_4\frac{d(w,Aw)d(Bw,Bp_{2n+1})+d(Ap_{2n+1},Bw)d(Bw,p_{2n+1})}{d(Aw,Bp_{2n+1})+d(p_{2n+1},Aw)}\\
&= sd(w,p_{2n+2})+sa_1d(w,p_{2n+1})\\
&\quad +sa_2\frac{d(w,AW)d(w,p_{2n+1})+d(p_{2n+1},p_{2n+2})d(p_{2n+1},Aw)}{d(w,p_{2n+2})+d(p_{2n+1},Aw)}\\
&\quad +sa_3\frac{d(Aw,p_{2n+2})d(w,Aw)+d(w,Ap_{2n+1})d(Ap_{2n+1},p_{2n+2})}{d(Aw,p_{2n+2})+d(p_{2n+1},Aw)}\\
&\quad +sa_4\frac{d(w,Aw)d(Bw,p_{2n+2})+d(p_{2n+2},Bw)d(Bw,p_{2n+1})}{d(Aw,p_{2n+2})+d(p_{2n+1},Aw)}.
\end{aligned}$$

Taking the limit as $n\to+\infty$, we get $d(w,Aw)<0$, hence $d(w,Aw)=0\to Aw=w$. Similarly, $Bw=w$. Now it is clear that w is a common fixed point of A and B.

Result 3 : Let us prove that A and B have a unique common fixed point.

Now consider l and m are two new common fixed point of A and B by equation (2), we get

$$\begin{aligned}
d(l,m) &= d(Al,Bm)\\
&\leq a_1d(l,m)+a_2\frac{d(l,A)d(l,Bm)+d(m,Bm)d(m,Al)}{d(lm)+d(m,Al)}\\
&\quad +a_3\frac{d(Al,Bm)d(l,Al)+d(l,Am)d(Am,Bm)}{d(Al,Bm)+d(m,mAl)}\\
&\quad +a_4\frac{d(l,Al)d(Bl,Bm)+d(Am,Bl)d(Bl,m)}{d(Al,Bm)+d(m,Al)}\\
&= a_1d(l,m).
\end{aligned}$$

Since, $a_1<1$, we get $d(l,m)=0$.

Hence we proved that A and B have a unique common fixed point in X.

Theorem 2 : Let (X,D) be a complete b-metric space with a coefficient $s\geq1$ and

$A, B : X \to X$ be two mapping on X, following conditions.

$$\begin{aligned} d(Ap, Bq) \quad \leq \quad & a_1 d(p,q) + a_2 \frac{d(q,Bq)[1+d(p,Ap)]}{1+d(p,q)} \\ & + a_3 d\frac{d(q,Bq)+d(q,Ap)}{1+d(q,Bq)d(q,Ap)} \\ & + a_4 d\frac{d(Aq,Bq)+d(Aq,Ap)+d(Bq,Ap)}{1+d(Aq,Bq)d(Aq,?Ap)d(Bq,Ap)} \\ & + a_5 d\frac{d(Bp,Aq)+d(Ap,Bq)+d(Ap,Bp)+d(Aq,Bq)}{1+d(Aq,Bq)d(Aq,Ap)d(Bq,Ap)} \end{aligned} \tag{8}$$

where all $p, q \in X$ and $a_1, a_2, a_3, a_4, a_5 \geq 0$. Also $s(a_1, a_2, a_3, a_4, a_5) < 1$. so that A and B have a unique common fixed point in X.

Proof : Let x_0 be arbitrary in X and a sequence (p_n) in X, then

$$p_{2n+1} = Ap_{2n}, p_{2n+2} = Bp_{2n+1} \text{ for all } n \in N. \tag{9}$$

Now consider $n \in N$ and $p_n = P_{n+1}$, if $n = 2_i$ and $p_{2i} - P_{2i+1}$, so that from the equation (8) and $p = p_{2i}, q = P_{2i+1}$, we get

$$\begin{aligned} d(p_{2+1} - P_{2i+1}) \quad = \quad & d(Ap_{2i}, Bp_{2i+1}) \\ & + a_2 \frac{d(p_{2i+1}, Bp_{2i+2})[1+(p_{2i}, Ap_{2i})]}{1+(p_{2i}, p_{2i+1})} \\ & + a_3 \frac{d(Ap_{2i}, Bp_{2i+1})d(p_{2i}, Ap_{2i})}{1+d(p_{2i+1}, Bp_{2i+1})+d(p_{2i+1}, Ap_{2i})} \\ & + a_4 \frac{d(Ap_{2i+1}, Bp_{2i+1})+d(Ap_{2i+1}, Ap_{2i})+d(Bp_{2i+1}, Ap_{2i})}{1+d(Ap_{2i+1}, Bp_{2i+1})+d(Ap_{2i+1}, Ap_{2i})d(Bp_{2i+1}, Ap_{2i})} \\ & + a_5 \frac{d(Bp_{2i}, Ap_{2i+1})+d(Ap_{2i}, Bp_{2i+1})+d(Ap_{2i}, Bp_{2i})+d(Ap_{2i+1}, Bp_{2i})}{1+d(Bp_{2i}, Ap_{2i+1})+d(Ap_{2i}, Bp_{2i+1})d(Ap_{2i}, Bp_{2i})+d(Ap_{2i+1}, Bp_{2i+1})} \\ = \quad & a_1 d(p_{2i}, p_{2i+1}) \\ & + a_2 \frac{d(p_{2i+1}, p_{2i+2})[1+(p_{2i}, p_{2i+2})]}{1+d(p_{2i}, p_{2i+1})} \\ & + a_3 \frac{d(p_{2i+1}, p_{2i+2})+d(p_{2i}, p_{2i+1})}{1+d(p_{2i+1}, p_{2i+2})d(p_{2i+1}, p_{2i+1})} \\ & + a_4 \frac{d(p_{2i+2}, p_{2i+2})+d(p_{2i+2}, p_{2i+1})+d(p_{2i+2}, p_{2i+1})}{1+d(p_{2i+2}, p_{2i+2})d(p_{2i+2}, p_{2i+1})+d(p_{2i+2}, p_{2i+1})} \\ & + a_5 \frac{d(p_{2i+1}, p_{2i+2})+d(p_{2i+1}, p_{2i+2})+d(p_{2i+1}, p_{2i+1})+d(p_{2i+2}, p_{2i+2})}{1+d(p_{2i+1}, p_{2i+2})d(p_{2i+1}, p_{2i+2})+d(p_{2i+1}, p_{2i+1})d(p_{2i+2}, p_{2i+2})}. \end{aligned}$$

Then

$$1 - (a_1 + a_2 + a_3 + a_4 + a_5)d(p_{2i+1}, p_{2i+2}) \leq 0.$$

Since, given constant is less than one, then we get $d(p_{2i+1}, p_{2i+2}) \leq 0$. Hence $p_{2i} = p_{2i+1} = p_{2i+2}$. Now it is clear that $p_{2i} = Ap_{2i} = Bp_{2i}$ such that p_{2i} is a common fixed point of A and B. If $n = 2i+1$ are some argument, then we get $p_{2i} = p_{2i+1}$.
Hence p_{2i+1} is also a common fixed point of A and B.
Now we consider that $p_n \neq p_{n+1}$ for all $n \in N$.
Result 1 : Let we have

$$d(p_n, p_{n+1}) \leq \frac{a_1}{1-(a_1+a_3+2a_4+2a^5)} d(p_{n-1}, p_n) \quad \text{for all} \quad n \in N \tag{10}$$

so that we can show that some terms in given condition.
Condition 1 : If $n = 2i+1$, from equatiion (8) and $p = p_{2i}$ and $q = p_{2i+1}$ we get

$$\begin{aligned}
d(p_{2i+1}, p_{2i+2}) &= d(Ap_{2i}, Bp_{2i+1}) \\
&\leq a_1 d(p_{2i}, p_{2i+1}) + a_2 \frac{d(p_{2i}, Bp_{2i+1})[1 + d(p_{2i}, Ap_{2i})]}{1 + d(p_{2i}, p_{2i+1})} \\
&\quad + a_3 \frac{d(p_{2i}, Bp_{2i+1}) + d(p_{2i}, Ap_{2i})}{1 + d(p_{2i+1}, Bp_{2i+1}) d(p_{2i+1}, Ap_{2i})} \\
&\quad + a_4 \frac{d(Ap_{2i+1}, Bp_{2i+1}) + d(Ap_{2i+1}, Ap_{2i}) + d(Bp_{2i+1}, Ap_{2i})}{1 + d(Ap_{2i+1}, Bp_{2i+1}) d(Ap_{2i+1}, Ap_{2i}) d(Bp_{2i+1}, Ap_{2i})} \\
&\quad + a_5 \frac{d(Bp_{2i}, Ap_{2i+1}) + d(Ap_{2i}, Bp_{2i+1}) + d(Ap_{2i}, Bp_{2i}) + d(Ap_{2i+1}, Bp_{2i+1})}{1 + d(Bp_{2i}, Ap_{2i+1}) d(Ap_{2i}, Bp_{2i+1}) d(Ap_{2i}, Bp_{2i}) + d(Ap_{2i+1}, Bp_{2i+1})} \\
&= a_1 d(p_{2i}, p_{2i+1}) \\
&\quad + a_2 \frac{d(p_{2i+1}, p_{2i+2})[1 + d(p_{2i}, p_{2i+1})]}{1 + d(p_{2i}, p_{2i+1})} \\
&\quad + a_3 \frac{d(p_{2i+1}, p_{2i+2}) + d(p_{2i}, p_{2i+1})}{1 + d(p_{2i+1}, p_{2i+2}) d(p_{2i+1}, p_{2i+1})} \\
&\quad + a_4 \frac{(p_{2i+2}, p_{2i+2}) + d(p_{2i+2}, p_{2i+1}) + d(p_{2i+2}, p_{2i+1})}{1 + d(p_{2i+2}, p_{2i+2}) d(p_{2i+2}, p_{2i+1}) + d(p_{2i+2}, p_{2i+1})} \\
&\quad + a_5 \frac{d(p_{2i+1}, p_{2i+2}) + d(p_{2i+1}, p_{2i+2}) + d(p_{2i+1}, p_{2i+1}) + d(p_{2i+2}, p_{2i+2})}{1 + d(p_{2i+1}, p_{2i+2}) d(p_{2i+1}, p_{2i+2}) + d(_{2i+1}, p_{2i+1}) d(p_{2i+2}, p_{2i+2})} \\
&= \frac{a_1}{1-(a_2+a_3+2a_4+2a_5)} d(p_{2i}, p_{2i+1}).
\end{aligned}$$

Hence we get

$$d(p_n, p_{n+1}) \leq \frac{a_1}{1-(a_2+a_3+2a_4+2a_5)} d(p_{n-1}, p_n), \quad n = 2i, \ i \in N. \tag{11}$$

Condition 2 : Let $n = 2i, i \in N$, now we apply some argument in condition (1), for $n = 2i$, we get

$$d(p_n, p_{n+1}) \leq \frac{a_1}{1-(a_2+a_3+2a_4+2a_5)} d(p_{n-1}, p_n), \quad n = 2i+1, \ i \in N. \tag{12}$$

By equation (11) and (12), we get

$$d(p_n, p_{n+1}) \leq \frac{a_1}{1-(a_2+a_3+2a_4+2a_5)} d(p_{n-1}, p_n), \quad \text{for all‘ } n \in N.$$

Hence $V = \frac{a_1}{1-(a_2+a_3+2a_4+2a_5)}$ as $V < \frac{1}{s} \leq 1$, because $S = (a_1 + a_2 + a3 + 2a_4 + 2a_5) < 1$, now it is clear that equation (10) hold.

By Lemma 2.1, we say that the sequence $\{p_n\}$ is a Cauchy sequence in (X, d), since (X, d) is a complete b-metric space and $\{p_n\}$ converges to some $u \in X$, as $n \to +\infty$.

Result 2 : Now we can prove that $Aw = Bw = W$. With the help of triangular inequality and equation (8), we get

$$\begin{aligned} d(w, Aw) &\leq s[d(w, p_{2n+2}) + d(p_{2n+2}, AW)] \\ &= sd(w, P_{2n+2}) + sd(w, Bp_{2n+1}) \\ &\leq sd(w, p_{2n+2}) + sa_1 d(w, p_{2n+1}) \\ &\quad + sa_2 \frac{d(p_{2n+1}, Bp_{2n+1})[1 + d(w, Aw)]}{1 + d(w, p_{2n+1})} \\ &\quad + sa_3 \frac{d(p_{2n+1}, Bp_{2n+1}) + d(p_{2n+1}, Aw)}{1 + d(p_{2n+1}, Bp_{2n+1}) + d(p_{2n+1}, Aw)} \\ &\quad + sa_4 \frac{d(Ap_{2n+1}, Bp_{2n+1}) + d(Ap_{2n+1}, Aw) + d(Bp_{2n+1}, Aw)}{1 + d(Ap_{2n+1}, Bp_{2n+1}) + d(Ap_{2n+1}, Aw)d(Bp_{2n+1}, Aw)} \\ &\quad + sa_5 \frac{d(Bw, Ap_{2n+1}) + d(Aw, Bp_{2n+1}) + d(Aw, Bw) + d(Ap_{2n+1}, Bp_{2n+1})}{1 + d(Bw, Ap_{2n+1})d(Aw, Bp_{2n+1}) + d(Aw, Bw)d(Ap_{2n+1}, Bp_{2n+1})} \end{aligned}$$

Taking the limit as $n \to +\infty$, we get

$$d(w, Aw) \leq (a_3 + 2a_4 + 2a_5)\ sd(w, Aw).$$

Since $a_3 + 2a_4 + 2a_5 < 1$, hence $d(w, Aw) = 0$. Then $Aw = w$. Similarly, we get

$$d(w, Bw) \leq 2sa_5\ d(w, Bw).$$

Since $2sa_5 < 1$, hence $d(w, Bw) = 0$. Then $Bw = w$.

Now, w is a common fixed point of A and B.

Result 3 : We can prove that A and B have a unique common fixed point. Consider that l and m are two new common fixed point of A and B. By equation (8), we get

$$\begin{aligned} d(l,m) &= d(Al,Bm) \\ &\leq a_1 d(l,m) + a_2 \frac{d(m,Bm)[1+(l,Al)]}{1+d(l,m)} \\ &\quad + a_3 \frac{d(m,Bm)+d(m,Al)}{1+d(m,Bm)+d(m,Al)} \\ &\quad + a_4 \frac{d(Am,Bm)+d(Am,Al)+d(Bm,Al)}{1+d(Am,Bm)d(Am,Al)d(Bm,Al)} \\ &\quad + a_5 \frac{d(Bl,Am)+d(Al,Bm)+d(Al,Bl)+d(Am,Bm)}{1+d(Bl,Am)d(Al,Bm)d(Al,Bl)d(Am,Bm)} \\ &\leq (a_1+a_3+2a_4+2a_5)d(l,m). \end{aligned}$$

Since $0 < a_1 + a_3 + 2a_4 + 2a_5 \leq 1$, we get $d(l,m) = 0$. Hence we prove that A and B have a unique common fixed point in X.

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