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# EFFICIENT WEAK ROMAN DOMINATION IN MYSCIELSKI GRAPHS

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**Abstract**

Let $G = (V, E)$ be a graph and $f$ be a function $f : V \to \{0, 1, 2\}$. A vertex $u$ with $f(u) = 0$ is said to be *undefended* with respect to $f$, if it is not adjacent to a vertex with positive weight. A $(2, 2) - packing$ is a function $f : V(G) \to \{0, 1, 2\}$ with $f(N[v]) \leq 2$ for all $v \in V(G)$. A vertex $v \in V_1 \cup V_2$ *influences* a set $S \subseteq N[v]$ with respect to a (2, 2) packing function $f : V \to \{0, 1, 2\}$ if for each $u \in S$, $f' : V \to \{0, 1, 2\}$ is such that $f'(v) = f(v) - 1, f'(u) = f(u) + 1,\ f'(w) = f(w)$ for every $w \in V - \{u, v\}$ leaves minimum number of undefended vertices in $N(v)$. We call such a set $S$, to be the *influence of* $v$, denoted by $I(v)$. The *weak Roman* influence of $f$ is defined to be $I_r(f) = \left| \bigcup_{v \in V_1 \cup V_2} I(v) \right|$ and the efficient weak Roman *domination number* is defined to be $F_r(G) = \max\{I_r(f) : f \text{ is a (2,2)-packing}\}$. In this paper we find the efficient weak Roman domination number of the Myscielski of paths and cycles.

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Key Words : *Domination number, Weak Roman domination number, Weak Roman influence.*

AMS Subject Classification : 05C69.

## 1. Introduction

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is a *dominating set* if every vertex $v \in V$ is an element of $S$ or adjacent to an element of $S$. The *domination number* $\gamma(g)$ of a graph $G$ is the minimum cardinality of a dominating set of $G$. The *open neighborhood* $N(v)$ of a vertex $v$ in a graph $G$ is the set of vertices that are adjacent to $v$. The *open neighborhood of a set* of vertices $S \subset V(G)$ is $N(S) = \bigcup\limits_{v \in S} N(v)$. The *closed neighborhood* $N[v]$ of a vertex $v$ is $N(v) \cup \{v\}$ and the *closed neighborhood of a set* of vertices $S \subset V(G)$ is $N[S] = N(S) \cup S$.

In a connected graph $G$, the *distance between two vertices* $u$ and $v$ is the number of edges in a shortest path joining $u$ and $v$ and is denoted by $d(u, v)$. A set $S$ is a *2-packing* if for any $u$ and $v$ in $S$, the distance $d(u, v) > 2$, or equivalently, if $|N[v] \cap S| \leq 1$ for every $v \in V(G)$.

Bange et al. [1] introduced the following efficiency measure for a graph $G$. The *efficient domination number* of a graph, denoted by $F(G)$, is the maximum number of vertices that can be dominated by a set $S$ that dominates each vertex at most once. A vertex $v$ of degree $deg(v) = |N(v)|$ dominates $|N[v]| = 1 + deg(v)$ vertices.

Grinstead and Slater [7] defined the *influence* of a set of vertices $S$ to be $I(S) = \sum\limits_{s \in S}(1 + deg(v))$, the total amount of domination being done by $S$. Because $S$ does not dominate any vertex more than once if and only if any two vertices in $S$ are at a distance at least three (that is, $S$ is a 2-packing), therefore, $F(G) = \max\{I(S) : S \text{ is a 2-packing}\}$. A set $S$ is an *efficient dominating set* if and only if $|N[v] \cap S| = 1$ for all vertices $v \in V(G)$, or equivalently, $S$ is an efficient dominating set if and only if $S$ is a 2-packing with $I(S) = n = F(G)$. A graph $G$ of order $n = |V(G)|$ has an efficient dominating set if and only if $F(G) = n$.

Cockanyne et al. [2] defined a *Roman dominating function* (RDF) in a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The *weight* of $f$ is $w(f) = \sum\limits_{v \in V} f(v)$. The *Roman domination number*, denoted by $\gamma_R(G)$ is the minimum weight of an RDF in $G$, that is $\gamma_R(G) = \min\{w(f) : f \text{ is an RDF in } G\}$. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-*function*. Roman domination has been studied in [2, 3, 4, 6, 9, 10, 12, 15, 16, 19].

Hedetniemi and Henning [9] defined the *weak Roman dominating function* as follows. For a function $f : V \to \{0, 1, 2\}$ let $V_0, V_1$ and $V_2$ be the sets of vertices assigned the values $0, 1$ and 2 respectively under $f$. A vertex $u \in V_0$ is undefended if it is not adjacent to a vertex in $V_1$ or $V_2$. The function $f$ is a weak Roman dominating function if each vertex $u \in V_0$ is adjacent to a vertex $v \in V_1 \cup V_2$ such that the function $f' : V \to \{0, 1, 2\}$ defined by $f'(u) = 1, f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. The *weak Roman domination number*, denoted by $\gamma_r(G)$ is the minimum weight of a weak Roman dominating function. That is, $\gamma_r(G) = \min\{w(f) : f$ is a weak Roman dominating function in $G\}$. Weak Roman domination has been studied in [17, 18].

Robert R. Rubalcaba and Peter J. Slater [13] extended the idea of efficiency to Roman domination as follows. A $(j, k) - packing$ is a function $f : V(G) \to \{0, 1, 2, \cdots, j\}$ with $f(N[v]) = \sum_{w \in N[v]} f(w) \leq k$ for all $v \in V(G)$. Thus, a 2-packing is a (1, 1)-packing, and in particular, a *(2, 2)-packing* is a function $f : V(G) \to \{0, 1, 2\}$ with $f(N[v]) \leq 2$ for all $v \in V(G)$. For a function $f : V(G) \to \{0, 1, 2\}$, the *Roman influence* of $f$, denoted by $I_R(f)$ is defined to be $I_R(f) = (|V_1| + |V_2|) + \sum_{v \in V_2} deg(v)$. The *efficient Roman domination number* of $G$, denoted by $F_R(G)$ is defined to be the maximum of $I_R(f)$ such that $f$ is a (2, 2)-packing. That is, $F_R(G) = \max\{I_R(f) : f$ is a (2, 2)-packing$\}$. A (2, 2)-packing $f$ with $F_R(G) = I_R(f)$ is called an $F_R(G)$- function. Graph $G$ is called *efficiently Roman dominatable* if $F_R(G) = n$ and when $F_R(G) = n$, the $F_R(G)$-function is called an *efficient Roman dominating function*.

Roushini Leely Pushpam and Kamalam [14] extended the idea of efficiency to weak Roman domination as follows. A vertex $v \in V_1 \cup V_2$ *influences* a set $S \subseteq N[v]$ with respect to a (2, 2) packing function $f : V \to \{0, 1, 2\}$ if for each $u \in S$, $f' : V \to \{0, 1, 2\}$ is such that $f'(v) = f(v) - 1$, $f'(u) = f(u) + 1$, $f'(w) = f(w)$ for every $w \in V - \{u, v\}$ leaves minimum number of undefended vertices in $N(v)$. We call such a set $S$, to be the influence of $v$, denoted by $I(v)$. The *weak Roman influence* of $f$ is defined to be $I_r(f) = \left| \bigcup_{v \in V_1 \cup V_2} I(v) \right|$ and the *efficient weak Roman domination number* is defined to be $F_r(G) = \max\{I_r(f) : f$ is a (2,2)-packing$\}$. If $F_r(G) = n$, then $G$ is said to be *efficiently weak Roman dominatable* or shortly EWRD and the corresponding (2, 2)-packing is called the *$F_r(G)$-function* of $G$.

For notation and graph theoretic terminology we in general follow [8]. Throughout this paper, we only consider simple, connected graphs. Let $G = (V, E)$ be a graph with vertex set $V$ and a subset $E$ of the unordered pairs of vertices, called edges.

## 2. Mycielski Graphs

In 1955, Mycielski [11] introduced an interesting graph transformation which transforms a graph $G$ into a new graph $\mu(G)$, called the *Mycielskian* of $G$. Using this construction, he created triangle-free graphs with large chromatic numbers. For a graph $G$ with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E$, the Mycielskian of $G$ is the graph $\mu(G)$, with vertex set $V \cup V' \cup \{w\}$, where $V' = \{u_i : v_i \in V\}$ and edge set $E \cup \{v_i u_j : v_i v_j \in E\} \cup \{u_i w : u_i \in V'\}$.

Mycielskians have many interesting properties concerning various kinds of parameters which was shown by Mycielski [11]. Fisher et al. [5] investigated Hamiltonicity, diameter, domination, packing and biclique partitions of Mycielskians.

In this section we characterize Mycielski graphs of paths and cycles which are EWRD.

**Theorem 1** : For paths $P_n, \mu(P_n)$ are EWRD.

**Proof** : Let $V(\mu(P_n)) = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\} \cup \{w\}$.

Define $f : V \to \{0, 1, 2\}$ as follows.

**Case 1** : $n$ is odd.

Let $f(w) = 1$ and $f(v_i) = \begin{cases} 1, & i \equiv 1, 2 \pmod{4}, \\ 0, & \text{ortherwise.} \end{cases}$

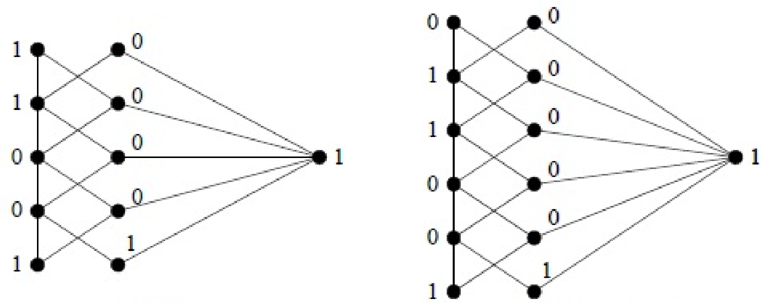
$f(u_i) = 0$ for all $i \neq n$ and $f(u_n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$

**Case 2** : $n$ is even.

Let $f(w) = 1$ and $f(v_i) = \begin{cases} 0, & \text{if } i \equiv 0,\ 1 \pmod 4, \\ 1, & \text{otherwise.} \end{cases}$

$f(u_i) = 0$ for all $i \neq n$ and $f(u_n) = \begin{cases} 1, & \text{if } n \equiv 2 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$

In both the cases, we see that $F_r(\mu(P_n)) = 2n + 1$. (Refer Figure 1). □




$F_r(\mu(P_5)) = 11$ $F_r(\mu(P_6)) = 13$

Figure 1: EWRD graphs with $F_r(\mu(P_n)) = 2n + 1$

**Theorem 2** : For cycles $C_n, \mu(C_n)$ are $EWRD$ if and only if $n \equiv 0, 3(mod\ 4)$.

**Proof** : Since $n \equiv 0, 3(mod\ 4)$. Define $f : V \to \{0, 1, 2\}$ by $f(w) = 1$,

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 1, 2 \pmod 4, \\ 1, & \text{otherwise.} \end{cases} \quad f(u_n) = \begin{cases} 1, & \text{if } n \equiv 3 \pmod 4, \\ 0, & \text{otherwise.} \end{cases}, \ f(u_i) = 0 \text{ for}$$

every $i \neq n$. Then $F_r(\mu(C_n)) = 2n + 1$. Hence $\mu(C_n)$ are $EWRD$.

Conversely let $F_r(\mu(C_n)) = 2n + 1$. We claim that $n \equiv 0, 3(mod\ 4)$. Suppose $n \equiv 2 \pmod 4$.

**Case 1** : $f(w) = 1$ and $f(u_i) = 1$, for some $i$.

Without loss of generality let $f(u_1) = 1$. Since $f$ is a $(2, 2)$-packing, $f(u_i) = 0$, for $i = 2, 3, \cdots, n$.

**Subcase (i)** : $f(v_1) = 1$.

Then $f(v_2) = 0$ and $f(v_3) = 0$. Since $f(v_3) = 0$, $f(v_4) = 1$. If $f(v_5) = 0$, then $v_5$ has to be defended by $v_6$, hence $f(v_6) = 1$. This implies that $f(N[u_5]) = 3$, a contradiction. Therefore $f(v_5) = 1$ which implies that $f(v_6) = 0$ and $f(v_7) = 0$. Since $f(v_7) = 0$, $f(v_8) = 1$. Suppose $f(v_9) = 0$ then $f(v_{10}) = 0$ which implies that $f(N[v_{10}]) = 3$, a contradiction. Therefore $f(v_9) = 1$. Proceeding this way, $f(v_{n-1}) = 0$ otherwise $f(N[v_n]) = 3$. Similarly $f(v_n) = 0$, otherwise $f(N[u_1]) = 3$. We see that $v_{n-1}$ is undefended.

**Subcase (ii)** : $f(v_1) = 0$.

Clearly $f(v_2) = 0$, otherwise $f(N[u_1]) = 3$. Therefore $f(v_n) \neq 1$, for otherwise $f(N[u_1]) = 3$. Since $f(v_n) \neq 1$, $v_1$ is undefended.

**Case 2** : $f(w) = 1$ and $f(u_i) = 0$ for every $i$.

**Subcase (i)** : $f(v_1) = 1$.

Then $f(v_3) = 0$. Suppose $f(v_2) = 0$, then $u_2$ will be undefended. Therefore $f(v_2) = 1$. Suppose $f(v_2) = 1$, then $f(v_4) = 0$, $f(v_5) = 1$. Since $f(v_5) = 1$, $f(v_7) = 0$ and $f(v_6) = 1$. Proceeding this way $f(v_{n-1}) = 1$. This shows that $f(N[u_n]) = 3$. Therefore $f(v_{n-1}) = 0$. But, now $v_{n-2}$ is undefended.

**Subcase (ii)** : $f(v_1) = 0$.

If $f(v_2) = 0$, then $f(v_n) = 1$ (for otherwise $v_1$ will be undefended) and $f(v_3) = 1$. Since $f(v_3) = 1$, $f(v_5) = 0$. Clearly $f(v_4) = 1$ as otherwise either $u_1$ or $u_3$ will be undefended. Since $f(v_4) = 1$, $f(v_6) = 0$. Clearly $f(v_7) = 1$. Proceeding this way we see that $f(v_{n-2}) = 1$ which implies $f(N[u_{n-1}]) = 3$. Suppose $f(v_2) = 1$ then $f(v_4) = 0$ and $f(v_n) = 0$. Then $f(v_3) = 1$ otherwise $u_1$ will be undefended. Hence $f(v_5) = 0$ and $f(v_6) = 1$. $f(v_6) = 1$ implies $f(v_8) = 0$. So $f(v_7) = 1$ for otherwise either $v_5$ or $v_7$ will become undefended. Proceeding this way $f(v_{n-1}) = 0$. Since $f(v_n) \neq 1$, $v_n$ becomes undefended.

**Case 3** : $f(w) = 0$ and $f(u_i) = 1$ for some $i$.

Without loss of generality let $f(u_1) = 1$ and let $f(u_i) = 0$ for $i = 2, 3, \cdots, n$.

Let $f(v_1) = 0$. If $f(v_2) = 1$ then $f(v_3) = 0$ in which case $u_2$ will be undefended. Therefore $f(v_2) = 0$. This implies that $f(v_n) = 1$, otherwise $v_1$ will be undefended. Clearly $f(v_3) = 1$, therefore $f(v_5) = 0$. $f(v_4) = 1$, otherwise $u_3$ will be undefended. Now $u_2$ or $u_4$ is undefended.

Suppose $f(v_1) = 1$, then $f(v_2) = 0$ and $f(v_3) = 0$. Clearly $f(v_4) = 1$. Now $v_3$ or $u_3$ is undefended.

**Case 4** : $f(w) = 0$, $f(u_i) = f(u_j) = 1$ for some $i, j$. Let $f(u_1) = f(u_2) = 1$. Then $f(u_i) = 0$ for every $i = 3, 4, \cdots, n$.

**Subcase (i)** : $f(v_1) = 1$.

Then $f(v_2) = 0$ and $f(v_3) = 0$. Therefore $f(v_4) = 1$ and $f(v_5) = 1$. So $f(v_6) = 0$, then $u_5$ is undefended.

**Subcase (ii)** : $f(v_1) = 0$.

Let $f(v_2) = 1$. Then $f(v_3) = 0$ and $f(v_4) = 0$. So $f(v_5) = 1$. Then either $u_4$ or $v_4$ is undefended. Suppose $f(v_2) = 0$. If $f(v_3) = 1$ then $f(v_4) = 0$. In this case $u_3$ is undefended. Therefore $f(v_3) = 0$. Then $f(v_n) = 1$. So $f(v_4) = f(v_5) = 1$. Therefore

$f(v_6) = 0$. Proceeding in the same manner $f(v_{n-1}) = 0$. Now $u_n$ is undefended. It can be similarly proved that for the other values of $i$ and $j$ some vertex in $V(\mu(C_n))$ is undefended.
In all the four cases we see that $F_r(\mu(C_n)) \neq 2n+1$ which is a contradiction. Therefore $k \not\equiv 2(\text{mod } 4)$. Similarly it can be shown that $k \not\equiv 1$ (mod 4). Therefore either $k \equiv 0(\text{mod } 4)$ or $k \equiv 3(\text{mod } 4)$. (Refer Figure 2). □

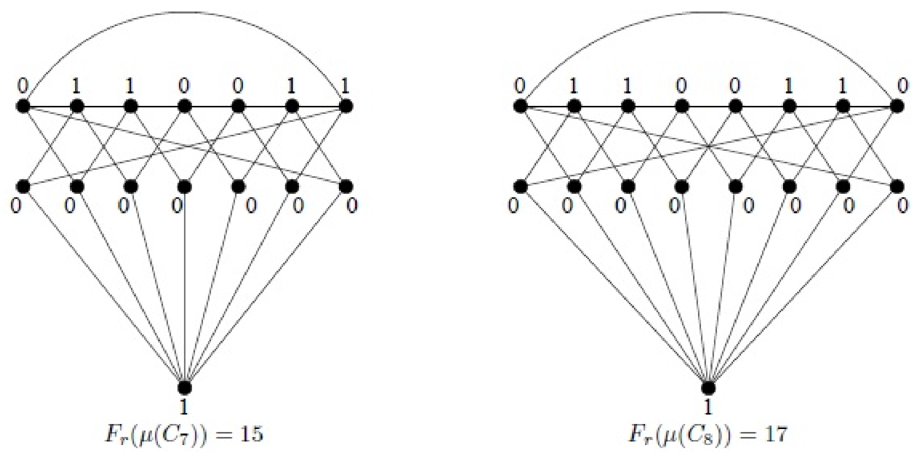



Figure 2: EWRD graphs with $F_r(\mu(C_n)) = 2n+1$

## 3. Conclusion

In this paper a study on efficiently weak Roman dominatable graphs has been initiated. Also we have obtained the efficient weak Roman domination number for certain Myscielski graphs. The concept of (2, 2)-packing used in this paper means that for any given vertex, the number of legions placed in its closed neighborhood does not exceed two. This would ensure that wastage in terms of placement of legions is minimized. This strategy would be very beneficial for companies engaged in logistics and service providing.

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