International J. of Pure & Engg. Mathematics (IJPEM)
ISSN 2348-3881, Vol. 3 No. II (August, 2015), pp. 101-109

FUNCTION OF ORDER BOUNDED VARIATION

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Abstract

Functions of bounded variation are a special class of functions with finite variation over an interval. Throughout this paper, we study the behavior of these functions and give some important theorems to show the essential properties of function of order bounded variation.

1. Introduction

Function of bounded variation is one of the basic concepts in mathematical analysis, which serves mathematics pure and applied. Two centuries ago, around 1881, Jordan (see [5]) introduced the notion of a function of bounded variation and established the relation between these functions and monotonic ones when he was studying convergence of Fourier series. Later on the concept of bounded variation was generalized in various directions by many mathematicians, such as L. Ambrosio, R. Caccioppoli, L. Cesari, E. Conway, G. Dal Maso, E. de Giorgi, S. Hudjaev, J. Musielak, O. Oleinik, W. Orlicz, F. Riesz, J. Smoller, L.Tonelli, A. Vol'pert, and N. Wiener, among many others. In 1970

Key Words : *Variation, Bounded variation, Total variation, Complete variation.*

2010 AMS Subject Classification : Primary 46B42.

A. N. Kolmogorov [2] has been studied in the book (Introductory Real Analysis). We divide this paper into two topics, which can be summarized as: we begin by giving some fundamental definitions and theorems needed for our topic. Also, we shall give some additional conditions named Lipschitz condition and prove some results concerning with it.

2. Function of o-bounded Variation

In this topic we explore function of order bounded variation and we discuss some important properties related with functions of order bounded variation.

M. Lind [4] discussed function of bounded variation and total variation definitions, and illustrative theorems to check whether or not the function is of bounded variation.

Let us define a function $f : [\alpha, \beta] \times [\alpha, \beta] \to R \times R$.

Definition 2.1 : A function f defined on an interval $[\alpha, \beta] \times [\alpha, \beta]$ is said to be of order bounded variation denoted by $O.B.V$, if there exists appositive constant M_1, M_2 such that

$$\sum_{k_1=1}^{n} \sum_{k_2=1}^{m} |f(x_{k_1}, y_{k_2}) - f(x_{k_2-1}, y_{k_2-1})| \leq M \tag{1}$$

where $M = (M_1, M_2)$, for every partition $(\alpha, \alpha) = (x_0, y_0), (x_1, y_1), \cdots, (x_n, y_m) = (\beta, \beta)$ of $[\alpha, \beta] \times [\alpha, \beta]$ of subdivision $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$.

Definition 2.2 : Suppose f be a function of o-bounded variation. The least upper bound of the sum (1) for all finite partition in a segment $[\alpha, \beta] \times [\alpha, \beta]$ is called the order total variation of the function f on the interval $[\alpha, \beta] \times [\alpha, \beta]$ and denoted by $O.T.V_f\{[\alpha, \beta] \times [\alpha, \beta]\}$. Thus

$$O.T.V_f\{[\alpha, \beta] \times [\alpha, \beta]\} = \sup \left\{ \sum_{k_1=1}^{n} \sum_{k_2-1}^{m} |f(x_{k_1}, y_{k_2}) - f(x_{k_2-1}, y_{k_2-1})| \right\}.$$

We need the following definition.

Definition 2.3 : Let $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_m)\}$ be a partition of an interval $[\alpha, \beta] \times [\alpha, \beta]$. Then the partition P' of $[\alpha, \beta] \times [\alpha, \beta]$ is a refinement of P if the partition points of P' include the partition points of P.

Now, we turn to some important properties of functions of bounded variation and their total variation but we first prove a theorem concerning refinements of partitions.

Theorem 2.4 : Suppose $f : [\alpha, \beta] \times [\alpha, \beta] \to R \times R$ be a function and P any partition of $[\alpha, \beta] \times [\alpha, \beta]$. If P' is any refinement of P then $O.B.V_f[P] \leq O.B.V_f[P']$.

Proof : Since any refinement of P can be obtained by adding points to P one at a time it's enough to prove the theorem in the case when we add just one point. Take $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_m)\}$ and add point (z, e) to a partition P and denote the result $P' = \{(x_0, y_0), (x_1, y_1), \cdots, (z, e), \cdots, (x_n, y_m)\}$. Assume that, $(x_{i-1}, y_{j-1}) < (z, e) < (x_i, y_j), (0 \leq i \leq n-1), (0 \leq j \leq m-1)$.
Then the triangle inequality given that

$$\begin{aligned} |f(x_i, y_i) - f(x_{i-1}, y_{j-1})| &= |f(x_i, y_j) - f(z, e) + f(z, e) - f(x_{i-1}, y_{j-1})| \\ &\leq (|f(x_i, y_j) - f(z, e)| + |f(z, e) - f(x_{i-1}, y_{j-1})|) \end{aligned}$$

and hence

$$\begin{aligned} &O.B.V_f[P] = \sum_{k_1=1}^{n} \sum_{k_2=1}^{m} |f(x_{k_1}, y_{k_2}) - f(x_{k_1-1}, y_{k_2-1})| \\ =& \left(|f(x_i, y_j) - f(x_{i-1}, y_{j-1})| + \sum_{\substack{k_1 = 1 \\ k_2 \neq i}}^{n} \sum_{\substack{k_2 = 1 \\ k_2 \neq j}}^{m} |f(x_{k_1}, y_{k_2}) - f(x_{k_1-1}, y_{k_2-1})| \right) \\ =& \ O.B.V_f[P']. \end{aligned}$$

Thus $O.B.V_f[P] \leq O.B.V_f[P']$. □

The above theory assures us that adding points to a partition P will only make the sum $O.B.V_p[f]$ larger or perhaps leave it unchanged.
We now state the following theorem.

Theorem 2.5 : Let f be $O.B.V$ on $[\alpha, \beta] \times [\alpha, \beta]$ and assume that $(a, b) \in [\alpha, \beta] \times [\alpha, \beta]$. Then f is $O.B.V$ on $[\alpha, a] \times [\alpha, b]$ and on $[a, \beta] \times [b, \beta]$ and we have

$$O.T.V_f\{[\alpha, \beta] \times [\alpha, \beta]\} = O.T.V_f\{[\alpha, a] \times [\alpha, b]\} + O.T.V_f\{[a, \beta] \times [b, \beta]\} \tag{2}$$

Proof : We consider a partition of interval $[\alpha, \beta] \times [\alpha, \beta]$ such that (a, b) is one of the points of subdivision, say $(x_{r_1}, y_{r_2}) = (a, b)$. Then

$$\begin{aligned} &\sum_{k_1=1}^{n} \sum_{k_2=1}^{m} |f(x_{k_1}, y_{k_2}) - f(x_{k_1-1}, y_{k_2-1})| = \sum_{k_1=1}^{r_1} \sum_{k_1=r_1+1}^{r_2} |f(x_{k_1-1}, y_{k_2-1})| \\ &+ \sum_{k_1=r_1+1}^{n} \sum_{k_2=r_2+1}^{n} |f(x_{k_1}, y_{k_2}) - f(x_{k_1-1}, y_{k_2-1})| \\ &\leq (O.T.V_f\{[\alpha, a] \times [\alpha, b]\} + O.T.V_f\{[a, \beta] \times [b, \beta]). \end{aligned} \tag{3}$$

Now consider an arbitrary partition of $[\alpha,\beta]\times[\alpha,\beta]$. It is clear that adding an extra point of subdivision to this partition can never decrease the sum

$$\sum_{k_1=1}^{n}\sum_{k_2=1}^{m}|f(x_{k_1},y_{k_2})-f(x_{k_1-1},y_{k_2-1})|.$$

Therefore (3) holds for any subdivision of $[\alpha,\beta]\times[\alpha,\beta]$ and hence

$$O.T.V_f\{[\alpha,\beta]\times[\alpha,\beta]\}\le O.T.V_f\{[\alpha,a]\times[\alpha,b]\}+O.T.V_f\{[a,\beta]\times[b,\beta]\}. \tag{4}$$

On the other hand, given $\epsilon_1,\epsilon_2>0$, there are partition of the intervals $[\alpha,a]\times[\alpha,b]$ and $[a,\beta]\times[b,\beta]$, respectively, such that

$$\sum_i\sum_j|f(x'_i,y'_i)-f(x'_{i-1},y'_{i-1})|>(O.T.V_f\{[\alpha,a]\times[\alpha,b]\}-\left(\frac{\epsilon_1}{2},\frac{\epsilon_2}{2}\right)$$

$$\sum_i\sum_j|f(x''_i,y''_j)-f(x''_{i'-1},y''_{j'-1})|>(O.T.V_f\{[a,\beta]\times[b,\beta]\}-\left(\frac{\epsilon_1}{2},\frac{\epsilon_2}{2}\right)$$

Combining all points of subdivision $(x'_i,y'_j),(x''_{i'},y''_{j'})$, we get a partition of the interval, with of subdivision $(x_{k_1},y-k_2)$, such that

$$\begin{aligned}\sum_{k_1}\sum_{k_2}|f(x_{k_1},y_{k_2})-f(x_{k_1-1},y_{k_1-1})| &= \left(\sum_i|f(x'_i,y'_j)-f(x'_{i-1},y'_{j-1})|\right)\\ &+\left(\sum_{i'}|f(x''_{i'},y''_{j'})-f(x''_{i'-1},y''_{j'-1})|\right)\\ &>\left((O.B.V_f\{[\alpha,a]\times\alpha,b]\}-\left(\frac{\epsilon_1}{2},\frac{\epsilon_2}{2}\right)\right)\\ &+\left((O.B.V_f\{[a,\beta]\times[b,\beta]\}-\left(\frac{\epsilon_1}{2},\frac{\epsilon_2}{2}\right)\right)\\ &=(O.T.V_f\{[\alpha,a]\times[\alpha,b]\}+O.T.V_f\{[a,\beta]\times[b,\beta]\}-(\epsilon_1,\epsilon_2)).\end{aligned}$$

Since $\epsilon_1,\epsilon_2>0$ is arbitrary, it follows that

$$O.T.V_f\{[\alpha,\beta]\times[\alpha,\beta]\}\ge O.T.V_f\{[\alpha,a]\times[\alpha,b]\}+O.T.V_f\{[a,\beta]\times[b,\beta]\}. \tag{5}$$

Comparing (4) and (5), we get (2). □

The following theorem proves that the function monotonic is $O.B.V$:

Theorem 2.6 : If f is monotonic on $[\alpha,\beta]\times[\alpha,\beta]$, then f is $O.B.V$ on $[\alpha,\beta]\times[\alpha,\beta]$.

Proof : Suppose f be increasing on $[\alpha,\beta]\times[\alpha,\beta]$. Then every partition $P=\{(x_0,y_0),(x_1,y_1),\cdots,(x_n,y_n)\}$ of $[\alpha,\beta]\times[\alpha,\beta]$ we have

$$\begin{aligned}&\sum_{k_1=1}^{n}\sum_{k_2=1}^{m}|f(x_{k_1},y_{k_2})-f(x_{k_1-1},y_{k_2-1})|\\&=\sum_{k_1=1}^{n}\sum_{k_2=1}^{m}(f(x_{k_1},y_{k_2})-f(x_{k_1-1},y_{k_2-1})=(f(\beta,\beta)-f(\alpha,\alpha)).\end{aligned}$$

Hence, f is $O.B.V.$ on $[\alpha,\beta]\times[\alpha,\beta]$.

Similarly, the function decreasing be $O.B.V.$ on $[\alpha,\beta]\times[\alpha,\beta]$. □

The converse of Theorem 1.7 is not true.

The following theorem proves that the $O.B.V$ is $O.B.$.

Theorem 2.7 : If a function f is $O.B.V$ on $[\alpha,\beta]\times[\alpha,\beta]$ then f is $O.B$ on $[\alpha,\beta]\times[\alpha,\beta]$.

Proof : Suppose $(x,y)\in[\alpha,\beta]\times[\alpha,\beta]$, consider partition P of $[\alpha,\beta]\times[\alpha,\beta]$, such that $P=\{(\alpha,\alpha),(x,y),(\beta,\beta)\}$. Since f is $O.B.V.$ on $[\alpha,\beta]\times[\alpha,\beta]$. Then

$$\sum_{k_1=1}^{n}\sum_{k_2=1}^{m}|f(x_{k_1},y_{k_2})-f(x_{k_1-1},y_{k_2-1})|=(|f(x,y)-f(\alpha,\alpha)|+|f(\beta,\beta)-f(x,y)|)\le M$$

where $M=(M_1,M_2)$.

Thus $|f(x,y)-f(\alpha,\alpha)|\le M$ which implies, $-M\le(f(x,y)-f(\alpha,\alpha))\le M$.

Since $\alpha<\beta$, put $-M=\alpha$ and $M=\beta$. Then

$$(\alpha,\alpha)\le(f(x,y)-f(\alpha\alpha))\le(\beta,\beta)$$

$$(\alpha,\alpha)+f(\alpha,\alpha)\le f(x,y)\le(\beta,\beta)+f(\alpha,\alpha).$$

So, $f(x,y)$ is o-bounded. □

The converse of Theorem 1.8 is not true.

Remark 2.8 : A function $f(x,y)$ defined on the whole real line $(-\infty,\infty)$ is said to be $O.B.V$ if there is a constants $M_1,M_2>0$, such that

$$O.T.V_f\{[\alpha,\beta]\times[\alpha,\beta]\}\le M$$

where $M=(M_1,M_2)$. For every pair of real number α and β $(\alpha<\beta)$. The quantity

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}}\quad O.T.V_f\{[\alpha,\beta]\times[\alpha,\beta]\}$$

is then called the total variation of f on $(-\infty, \infty)$ denoted by $O.T.V_f(-\infty, \infty)$.

Remark 2.9 : Suppose f is a function of $O.V.B$, then

$$O.T.V_{((c_1,c_2)f)}\{[\alpha,\beta]\times[\alpha,\beta]\} = |(c_1,c_2)|O.T.V_f\{[\alpha,\beta]\times[\alpha,\beta]\} \tag{6}$$

For any constant (c_1, c_2).

The following theorem tell us that the sum of two $O.B.V$ are $O.B.V$.

Theorem 2.10 : If $f(x, y)$ and $g(x, y)$ and are function of $O.B.V$ on $[\alpha, \beta] \times [\alpha, \beta]$, then so is $f + g$, and

$$O.T.V_{f+g}\{[\alpha,\beta]\times[\alpha,\beta]\} \leq O.T.V_f\{[\alpha,\beta]\times[\alpha,\beta]\} + O.T.V_g\{[\alpha,\beta]\times[\alpha,\beta]\}. \tag{7}$$

Proof : For any partition of the interval $[\alpha, \beta] \times [\alpha, \beta]$, we have

$$\begin{aligned}
&\sum_{k_1}\sum_{k_2} |f(x_{k_1}, y_{k_2}) + g(x_{k_1}, y{+}_{k_2}) - f(x_{k_1-1}, y_{k_2-1}) - g(x_{k_1-1}, y_{k_2-1})| \\
&\leq \left(\sum_{k_1}\sum_{k_2} |f(x_{k_1}, y_{k_2}) - f(x_{k_1-1}, y_{k_2-1})|\right. \\
&\left. + \sum_{k_1}\sum_{k_2} |g(x_{k_1}, y_{k_2}) - g(x_{k_1-1}, y_{k_2-1})|\right).
\end{aligned}$$

Taking the least upper bound of both sides over all partitions of $[\alpha, \beta] \times [\alpha, \beta]$, and nothing that theorem 11 [1], we have

$$O.T.V_{f+g}\{[\alpha,\beta]\times[\alpha,\beta]\} \leq O.T.V_f\{[\alpha,\beta]\times[\alpha,\beta]\} + O.T.V_g\{[\alpha,\beta]\times[\alpha,\beta]\}).$$

□

Remark 2.11 : It follows from (8) and (9) that any linear combination of function of $O.B.V$ is itself a function of $O.B.V$. In other words, the set of all functions of o-bounded variation on a given interval is a linear space.

Corollary 2.12 : The function

$$v(x, y) = O.T.V_f\{[\alpha, x] \times [\alpha, y]\} \tag{8}$$

is non-decreasing.

Proof : For any $(x_1, y_1) < (x_2, y_2)$ we have

$$(v(x_2, y_2) - v(x_1, y_1)) = O.T.V_f\{[x_1, x_2] \times [y_1, y_2]\} \geq (0, 0).$$

The total variation of any function of o-bounded variation on any order interval is nonnegative. Then

$$(v(x_2, y_2) - v(x_1, y_1)) \geq (0, 0).$$

Thus

$$v(x_2, y_2) \geq v(x_1, y_1).$$

Thus function $v(x, y)$ is non-decreasing. □

We use the Jordan theorem 2.1.7 [4] to prove the following theorem.

Theorem 2.13 : Suppose $f(x, y)$ is $O.B.V$ if and only if $f(x, y)$ is the difference of two non-decreasing functions.

Proof : Assume that $f(x, y)$ is $O.B.V$ and let $v(x, y) = V\{[\alpha, x] \times [\alpha, y]\}$, $(x, y) \in [\alpha, \beta] \times [\alpha, \beta]$ and $v(\alpha, \alpha) = 0$.

Then, clearly $f(x, y) = v(x, y) - [v(x, y) - f(x, y)]$.

By Corollary 2.12, we have $v(x, y)$ is non-decreasing. Now, we will show $v(x, y) - f(x, y)$ is non-decreasing.

For any $(x_1, y_1) < (x_2, y_2)$,

$$\begin{aligned}
&(f(x_2, y_2) - f(x_1, y_1)) \leq |f(x_2, y_2) - f(x_1, y_1)| \\
&\quad \leq O.T.V_f\{[x_1, x_2] \times [y_1, y_2]\} \\
&\quad = (v(x_2, y_2) - v(x_1, y_1)) \\
&\Rightarrow (f(x_2, y_2) - f(x_1, y_1)) \leq (v(x_2, y_2) - v(x_1, y_1))
\end{aligned}$$

Thus $(v(x_1, y_1) - f(x_1, y_1)) \leq (v(x_2, y_2) - f(x_2, y_2))$.

So, $v(x, y) - f(x, y)$ is non-decreasing.

Conversely, suppose that $f(x, y) = g(x, y) - h(x, y)$ with g and h non-decreasing. Since h is non- decreasing, $-h$ is decreasing [3] and thus $f(x, y) = g(x, y) + (-h(x, y))$ is the sum of two monotone functions, so theorem 2.7 together with theorem 2.11 gives that $f(x, y)$ is $OB.V$.

Theorem 2.14 : If a function $f(x, y)$ is of $O.B.V$, then it has a derivative almost everywhere.

Proof : By Lebesgue theorem, a monotonic function $f(x, y)$ has a finite derivative almost everywhere.

So a monotonic function $f(x, y)$ is of $O.B.V$ by Theorem 2.6.

Thus, a function $f(x, y)$ has a derivative almost everywhere. $\square$

Remark 2.15 : The space of function $O.B.V$ on $[\alpha, \beta] \times [\alpha\beta]$ is equipped with the norm

$$\|f(x, y)\| = |f(\alpha, \alpha)| + O.T.V_f\{[\alpha, \beta] \times [\alpha, \beta]\}.$$

3. Condition for Order Bounded Variation

In this topic we shall give some additional conditions which will guarantee that a function is of order bounded variation.

Definition 3.1 : Suppose $[\alpha, \beta] \times [\alpha, \beta] \to R \times R$, f is said to satisfy a Lipschitz condition if there exists a positive constants M_1, M_2 such that for every $(x, y), (x', y') \in [\alpha, \beta] \times [\alpha, \beta]$, we have

$$|f(x, y) - f(x', y')| \leq M|(x, y) - (x', y')|$$

where $M = (M_1, M_2)$.

Theorem 3.2 : If $f : [\alpha, \beta] \times [\alpha, \beta] \to R \times R$ satisfies a Lipschitz condition on $[\alpha, \beta] \times [\alpha, \beta]$ with constant c_1, c_2, then f is $O.B.V$ and

$$O.T.V_f\{[\alpha, \beta] \times [\alpha, \beta]\} \leq (c_1, c_2)((\beta, \beta) - (\alpha, \alpha)).$$

Proof : Suppose that $|f(x, y) - f(x', y')| \leq (c_1, c_2|(x, y) - (x', y')|$ for every $(x, y)(x', y') \in [\alpha, \beta] \times [\alpha, \beta]$.

Take an arbitrary partition $P = \{(x_0, y_0), \cdots, (x_n, y_m)\}$ of $[\alpha, \beta] \times [\alpha, \beta]$. Then

$$\begin{aligned} O.B.V_f[P] &= \sum_{k_1=1}^{n} \sum_{k_2=1}^{m} |f(x_{k_1}, y_{k_2}) - f(x_{k_1-1}, y_{k_2-1})| \\ &\leq \sum_{k_1=1}^{n} \sum_{k_2=1}^{m} (c_1, c_2)|(x_{k_1}, y_{k_2}) - (x_{k_1-1}, y_{k_2-1})| \\ &= (c_1, c_2)((\beta, \beta) - (\alpha, \alpha)). \end{aligned}$$

Since P was arbitrary, the inequality above is valid for any partition, which means that the sums $O.B.V_f[P]$ are bounded above by $(c_1, c_2)((\beta, \beta) - (\alpha, \alpha))$ whence it follows that f is $O.B.V$ on $[\alpha, \beta] \times [\alpha, \beta]$ and $O.T.V_f\{[\alpha, \beta] \times [\alpha, \beta]\} \leq (c_1, c_2)((\beta, \beta) - (\alpha, \alpha))$. $\square$

Theorem 3.3 : If $[\alpha,\beta] \times [\alpha,\beta] \rightarrow R \times R$ is differentiable on $[\alpha,\beta[\times]\alpha,\beta[$ and that $f'(x,y)$ is $O.B.$ on that interval, then $f(x,y)$ is $O.B.V$ on $[\alpha,\beta] \times [\alpha,\beta]$ and

$$O.T.V_f\{[\alpha,\beta] \times [\alpha,\beta]\} \leq (M_1, M_2)((\beta,\beta) - (\alpha,\alpha)).$$

Proof : For any $(x,y),(x',y') \in]\alpha,\beta[\times]\alpha,\beta[$ we have

$$(f(x,y) = f(x',y')) = f'(c_1,c_2)((x,y) - (x',y'))$$

for some (c_1,c_2) between (x,y) and (x',y') according to the mean value heorem [1], since $f'(x,y)$ is o-bounded, then $(\alpha,\alpha) \leq f'(x,y) \leq (\beta,\beta)$. Hence

$$\frac{f(x,y) - f(x',y')}{((x,y) - (x',y'))} = f'(c_1,c_2).$$

Since $(\alpha,\alpha) \leq f'(c_1,c_2) \leq (\beta,\beta)$, thus

$$(\alpha,\alpha) \leq \frac{f(x,y) - f(x',y')}{((x,y) - (x',y'))} \leq (\beta,\beta)$$

$$\frac{f(x,y) - f(x',y')}{((x,y) - (x',y'))} \leq (\beta,\beta) \Rightarrow \left|\frac{f(x,y) - f(x',y')}{((x,y) - (x',y'))}\right| \leq |(\beta,\beta)|.$$

Then $|f(x,y) - f(x',y')| \leq |(\beta,\beta)||(x,y) - (x',y,)$, for any $(x,y),(x',y') \in [\alpha,\beta] \times [\alpha,\beta]$, that is $f(x,y)$ satisfies a lipschitz condition on $[\alpha,\beta] \times [\alpha,\beta]$ with constant $|(\beta,\beta)|$ and thus Theorem 2.3 gives the result. □

References

[1] Khateeb A., Al-Qawasmi O., Hamamde J. and Al-Muhtaseb K., Functions of bounded variation and Riemann-Stieltjes integral, The Second Students Innovative Conference (SIC2013).

[2] Kolmogorov A. N. and Fomin S. V., Introductory Real Analysis, Dover Publications, Inc., New York, (1970).

[3] Van Derwalt J¿ H., Order Convergence on Archimedean Vector Lattice and Application, University of Pretoria etd., (2006).

[4] Lind M., Function of Bounded Variation, Mathematics C-Level thesis, Karlstadsuniversitet, (2006).

[5] Jordan C, Sur la série de Fourier, Comptes Rendusdel' Académie des Sciences, Paris, 2 (1881), 228230. View at Google Scholar.