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EXTENDED INTUITIONISTIC FUZZY METRIC AND FIXED POINT THEOREMS

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Abstract

In this paper, the notion of extended intuitionistic fuzzy metric space is introduced and prove some fixed point theorems of contraction mapping of Banach type with respect to the extended intuitionistic fuzzy metric.

1. Introduction

The notion of fuzzy set was introduced by Zadeh [19] in 1965 whose fundamental component is only a degree of membership. In 1986, Atanassov [2, 3, 4], generalized this concept into intuitionistic fuzzy sets using a degree of membership and non-membership

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under the condition that the sum of these two degrees does not exceed one.

In 1975, Kramosil and Michalek [11] introduced the concept of fuzzy metric spaces. It is worth noticing that there exist at least five different concepts of a fuzzy metric space (see Artico and Moresco [1], Deng [5], George and Veeramani [6], Erceg [8], Kaleva and Seikkala [12], Kramosil and Michalek [11]. In 1989, Grabiec [9] proved an analog of the Banach contraction theorem in fuzzy metric spaces (in the sense of Kramosil and Michalek [11]). In his proof, he used a fuzzy version of Cauchy sequence. It is worth noticing that in the literature in order to prove fixed point theorems in fuzzy metric space, authors used two different types of Cauchy sequences. For details see [18]. The existence of fixed points for maps in fuzzy metric spaces was studied by many authors; see e.g. Gregori and Sapena [10], Mihe [14]. In 2014, R. Plembaniak [16] introduced the concept of generalized fuzzy metric space. In such ways, George and Veeramani [7], Park [15] has defined intuitionistic fuzzy metric spaces and obtained several classical theorems on this new structure. In 2008, R. Saadati et.al [17] introduced Modified intuitionistic fuzzy metric space. R.Krishnakumar and D.Dhamodharan [13] introduced B2-metric space.

In this paper, the concept of extended intuitionistic fuzzy metric is introduced which is the generalization of an intuitionistic fuzzy metric. Next, we discussed the G contraction and GV contraction of Banach type with respect to the extended intuitionistic fuzzy metric (in sense of Grabiec M and Gregori V, et al. respectively). Moreover, we provide the condition for guaranteeing the existence of a fixed point theorem for these single-valued contractions.

2. Preliminaries

Definition 2.1 : A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $a * 1 = a$ for all $a \in [0,1]$;

(d) $a * b \leq c * d$ whenever $a \leq c$ and $c \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2.2 : A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if

$\diamond$ is satisfying the following conditions:
(a) $\diamond$ is commutative and associative;
(b) $\diamond$ is continuous;
(c) $a \diamond 0 = a$ for all $a \in [0, 1]$;
(d) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $c \leq d$, and $a, b, c, d \in [0, 1]$.

Definition 2.3 : The 5-tuple $(X, M, N, *, \diamond)$ is said to be intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M,N are intuitionistic fuzzy sets on $X_2 \times [0, \infty)$ satisfying the following conditions:

1. $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;
2. $\forall_{x,y \in X}\{M(x, y, 0) = 0\}$;
3. $\forall_{x,y \in X}\{\forall_{t>0}\{M(x, y, t) = 1\} \Leftrightarrow x = y\}$;
4. $\forall_{x,y \in X}\forall_{t>0}\{M(x, y, t) = M(y, x, t)\}$;
5. $\forall_{x,y,z \in X}\forall_{t,s>0}\{M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)\}$;
6. $M(x, y, .) : [0, \infty) \to [0, 1]$ is left-continuous, for all $x, y \in X$;
7. $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;
8. $\forall_{x,y \in X}\{N(x, y, 0) = 1\}$;
9. $\forall_{x,y \in X}\{\forall_{t>0}\{N(x, y, t) = 0\} \Leftrightarrow x = y\}$;
10. $\forall_{x,y \in X}\forall_{t>0}\{N(x, y, t) = N(y, x, t)\}$;
11. $\forall_{x,y,z \in X}\forall_{t,s>0}\{N(x, z, t + s) \leq N(x, y, t) \diamond N(y, z, s)\}$;
12. $N(x, y, .) : [0, \infty) \to [0, 1]$ is right-continuous, for all $x, y \in X$;
13. $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

3. Main Result

Definition 3.1 : Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. The functions (M,N) is said to be a G-Extended intuitionistic fuzzy metric on X if the following conditions hold:

1. $\forall_{x,y,z \in X} \forall_{t,s>0} \{M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)\}$;

2. $M(x, y, .) : [0, 1) \to [0, 1]$ is left-continuous, for all $x, y \in X$;

3. for any sequences $(x_m : m \in \mathsf{N})$ and $(y_m : m \in \mathsf{N})$ in X such that

$$\forall_{t>0} \forall_{p \in \mathsf{N}} \left\{ \lim_{m \to \infty} M(x_m, x_{m+p}, t) = 1 \right\} \tag{1}$$

and

$$\forall_{t>0} \left\{ \lim_{m \to \infty} M(x_m, y_m, t) = 1 \right\}, \tag{2}$$

we have

$$\forall_{t>0} \left\{ \lim_{m \to \infty} M(x_m, y_m, t) = 1 \right\}. \tag{3}$$

4. $\forall_{x,y,z \in X} \forall_{t,s>0} \{N(x, z, t + s) \leq N(x, y, t) \diamond N(y, z, s)\}$;

5. $N(x, y, .) : [0, 1) \to [0, 1]$ is right-continuous, for all $x, y \in X$;

6. for any sequences $(x_m : m \in \mathsf{N})$ and $(y_m : m \in \mathsf{N})$ in X such that

$$\forall_{t>0} \forall_{p \in \mathsf{N}} \left\{ \lim_{m \to \infty} N(x_m, x_{m+p}, t) = 0 \right\} \tag{4}$$

and

$$\forall_{t>0} \left\{ \lim_{m \to \infty} N(x_m, y_m, t) = 0 \right\}, \tag{5}$$

we have

$$\forall_{t>0} \left\{ \lim_{m \to \infty} N(x_m, y_m, t) = 0 \right\}. \tag{6}$$

Remark 3.1 : If $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy metric space, then the intuitionistic fuzzy metric M is a G-Extended intuitionistic fuzzy metric on X. However, there exists a G-Extended intuitionistic fuzzy metric on X which is not an intuitionistic fuzzy metric on X.

Definition 3.2 : A sequence $(x_m : m \in \mathsf{N})$ in X is N-Cauchy in Grabiec$'s$ sense (we say N-G-Cauchy) if

$$\forall_{t>0}\forall_{p\in\mathsf{N}}\left\{\lim_{m\to\infty} M(x_m, x_{m+p}, t) = 1\right\}$$

and

$$\forall_{t>0}\forall_{p\in\mathsf{N}}\left\{\lim_{m\to\infty} N(x_m, x_{m+p}, t) = 0\right\}.$$

Definition 3.3 : A sequence $(x_m : m \in \mathsf{N})$ in X is N-convergent to $x \in X$ if

$$\forall_{t>0}\left\{\lim_{m\to\infty} M(x_m, x, t) = 1\right\}$$

and

$$\forall_{t>0}\left\{\lim_{m\to\infty} N(x_m, x, t) = 0\right\}.$$

Definition 3.4 : An intuitionistic fuzzy metric space is called N-G-complete if each N-G-Cauchy sequence $(x_m : m \in N)$ in X is N-convergent to some $x \in X$ and

$$\forall_{t>0}\left\{\lim_{m\to\infty} M(x_m, x, t) = \lim_{m\to\infty} M(x, x_m, t) = 1\right\}$$

and

$$\forall_{t>0}\left\{\lim_{m\to\infty} N(x_m, x, t) = \lim_{m\to\infty} N(x, x_m, t) = 0\right\}$$

.

Lemma 3.1 : Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and let the map (M, N) be a G-Extended intuitionistic fuzzy metric on X. Then for each $x, y \in X$ the following property holds :

$$\left\{\forall_{t>0}\left\{M(x, y, t) = 1 \wedge M(y, x, t) = 1\right\} \Rightarrow \{x = y\}\right\}$$

and

$$\left\{\forall_{t>0}\left\{N(x, y, t) = 0 \vee N(y, x, t) = 0\right\} \Rightarrow \{x = y\}\right\}.$$

Proof : Let $x, y \in X$ such that

$$\forall_{t>0}\left\{M(x,y,t)=1 \wedge M(y,x,t)=1\right\} \tag{7}$$

and

$$\forall_{t>0}\left\{N(x,y,t)=0 \vee N(y,x,t)=0\right\} \tag{8}$$

be arbitrary and fixed. Using Definition 3.1 , we get

$$\forall_{t>0}\left\{M(x,y,t) \geq M\left(x,y,\frac{t}{2}\right) * M\left(y,x,\frac{t}{2}\right) = 1 * 1 = 1\right\} \tag{9}$$

$$\forall_{t>0}\left\{N(x,y,t) \leq N\left(x,y,\frac{t}{2}\right) \diamond N\left(y,x,\frac{t}{2}\right) = 0 \diamond 0 = 0\right\} \tag{10}$$

Defining the sequences $(x_m = x : m \in \mathsf{N})$ and $(y_m = y : m \in \mathsf{N})$,
From (7), (8), (9) and (10), we have

$$\forall_{t>0}\forall_{p\in\mathsf{N}}\, M\left\{\lim_{m\to\infty} M(x_m, x_{m+p}, t) = 1\right\}$$

$$\forall_{t>0}\forall_{p\in\mathsf{N}}\, N\left\{\lim_{m\to\infty} N(x_m, x_{m+p}, t) = 0\right\}$$

and

$$\forall_{t>0}\left\{\lim_{m\to\infty} M(x_m, y_m, t) = 1\right\}$$

$$\forall_{t>0}\left\{\lim_{m\to\infty} N(x_m, y_m, t) = 0\right\}$$

Hence, the properties (3) and (6) in Definition 3.1 is hold.
We see that

$$\forall_{t>0}\left\{\lim_{m\to\infty} M(x_m, y_m, t) = 1\right\}$$

$$\forall_{t>0}\left\{\lim_{m\to\infty} N(x_m, y_m, t) = 0\right\}$$

which, by the definition of the sequences $(x_m = x : m \in \mathsf{N})$ and $(y_m = y : m \in \mathsf{N})$, gives

$$\forall_{t>0}\left\{M(x,y,t) = 1\right\}$$

$$\forall_{t>0}\Big\{N(x,y,t)=0\Big\}$$

Hence, we conclude that x = y. □

Theorem 3.1 : Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, and let (M, N) be a G-extended intuitionistic fuzzy metric on X such that

$$\forall_{x,y\in X}\Big\{\lim_{t\to\infty} M(x,y,t)=1\Big\} \tag{11}$$

and

$$\forall_{x,y\in X}\Big\{\lim_{t\to\infty} N(x,y,t)=0\Big\} \tag{12}$$

.

Let $T : X \to X$ be an N-G-contraction of Banach type, i.e., T is a mapping satisfying

(B1) $\exists_{k\in(0,1)}\forall_{x,y\in X}\forall_{t>0}\Big\{M(T(x),T(y),kt)\geq M(x,y,t)\Big\}$ and

(B2) $\exists_{k\in(0,1)}\forall_{x,y\in X}\forall_{t>0}\Big\{N(T(x),T(y),kt)\leq N(x,y,t)\Big\}$.

We assume that an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is N-G-complete. Then T has a unique fixed point $w \in X$, and for each $x \in X$, the sequence $(x_m = T^m(x_0) : x_0 = x, m \in N)$ is convergent to w. Moreover, $M(w, w, t) = 1$,$N(w, w, t) = 0$ for all $t > 0$.

Proof : The proof of this theorem is divided into four steps.

Step 1 : It is easy to see that for each $x \in X$ the sequence $(x_m = T^m(x_0) : x_0 = x, m \in \mathsf{N})$ satisfies

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{m\in\mathsf{N}}\left\{M(x_m,x_{m+1},kt)\geq M\left(x_0,x_1,\frac{t}{k^{m-1}}\right)\right\} \tag{13}$$

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{m\in\mathsf{N}}\left\{N(x_m,x_{m+1},kt)\leq N\left(x_0,x_1,\frac{t}{k^{m-1}}\right)\right\} \tag{14}$$

Indeed, let $x_0 = x \in X$ be arbitrary and fixed and let $(x_m = T^m(x_0) : m \in \mathsf{N})$. Let

$k \in (0,1)$ be as in $(B1)$, $m \in \mathsf{N}$ and $t > 0$ be arbitrary and fixed. From $(B1)$ We get

$$\begin{aligned} M(x_m, x_{m+1}, kt) &= M(T(x_{m-1}), T(x_m), kt) \geq M(x_{m-1}, x_m, t) \\ &= M\left(T(x_{m-2}), T(x_{m-1}), k\frac{t}{k}\right) \geq M\left(x_{m-2}, x_{m-1}, \frac{t}{k}\right) \\ &= M\left(T(x_{m-3}), T(x_{m-2}), k\frac{t}{k^2}\right) \\ &\geq M\left(x_{m-3}, x_{m-1}, \frac{t}{k^2}\right) \\ &\geq .... \geq M\left(x_0, x_1, \frac{t}{k^{m-1}}\right). \end{aligned}$$

Hence, the property (13) holds.

Similarly, let $(x_m = T^m(x_0) : m \in \mathsf{N})$. Let $k \in (0,1)$ be as in $(B2)$, $m \in \mathsf{N}$ and $t > 0$ be arbitrary and fixed. From $(B2)$ We get

$$\begin{aligned} N(x_m, x_{m+1}, kt) &= N(T(x_{m-1}), T(x_m), kt) \leq N(x_{m-1}, x_m, t) \\ &= N\left(T(x_{m-2}), T(x_{m-1}), k\frac{t}{k}\right) \leq N\left(x_{m-2}, x_{m-1}, \frac{t}{k}\right) \\ &= N\left(T(x_{m-3}), T(x_{m-2}), k\frac{t}{k^2}\right) \\ &\leq N\left(x_{m-3}, x_{m-1}, \frac{t}{k^2}\right) \\ &\leq .... \leq N\left(x_0, x_1, \frac{t}{k^{m-1}}\right). \end{aligned}$$

Hence, the property (14) holds.

Step 2 : We see that for each $x \in X$ the sequence $(x_m = T^m(x_0) : x_0 = x, m \in \mathsf{N})$ is N-G-Cauchy, i.e., it satisfies

$$\forall_{t>0} \forall_{p \in \mathsf{N}} \left\{ \lim_{m \to \infty} M(x_m, x_{m+p}, t) = 1 \right\} \tag{15}$$

and

$$\forall_{t>0} \forall_{p \in \mathsf{N}} \left\{ \lim_{m \to \infty} N(x_m, x_{m+p}, t) = 0 \right\}$$

Indeed, let $x_0 = x \in X$ be arbitrary and fixed and let $(x_m = T^m(x) : m \in \mathsf{N})$. Let $m, p \in \mathsf{N}$ and $t > 0$ be arbitrary and fixed. Then

$$\begin{aligned} M(x_m, x_{m+p}, t) &\geq M\left(x_m, x_{m+1}, \frac{t}{p}\right) * ..^{(p)}.. * M\left(x_{m+p-1}, x_{m+p}, \frac{t}{p}\right) \\ &\geq M\left(x_0, x_1, \frac{t}{pk^m}\right) * ..^{(p)}.. * M\left(x_0, x_1, \frac{t}{pk^{m+p-1}}\right) \end{aligned}$$

$$
\begin{aligned}
N(x_m, x_{m+p}, t) &\leq N\left(x_m, x_{m+1}, \frac{t}{p}\right) \diamond ..^{(p)}.. \diamond N\left(x_{m+p-1}, x_{m+p}, \frac{t}{p}\right) \\
&\leq N\left(x_0, x_1, \frac{t}{pk^m}\right) \diamond ..^{(p)}.. \diamond N\left(x_0, x_1, \frac{t}{pk^{m+p-1}}\right)
\end{aligned}
$$

Now, using (11), (12) we obtain

$$\lim_{m\to\infty} M(x_m, x_{m+p}, t) \geq 1 * ..^{(p)}. * 1 = 1.$$

$$\lim_{m\to\infty} N(x_m, x_{m+p}, t) \leq 0 \diamond ..^{(p)}. \diamond 0 = 0.$$

Thus (15) holds.

Step 3 : Next we see that for each $x \in X$ the sequence $(x_m = T^m(x_0) : x_0 = x, m \in \mathsf{N})$ is convergent to a fixed point of T.

Indeed, let $x_0 = x \in X$ be arbitrary and fixed and let $(x_m = T^m(x_0) : m \in \mathsf{N})$. By Step 2 the sequence $(x_m : m \in \mathsf{N})$ is N-G-Cauchy in X. By the N-G-completeness of X, there exists $w \in X$ such that $(x_m : m \in \mathsf{N})$ is N-convergent to w ($i.e., \forall_{t>0}\{\lim_{m\to\infty} M(x_m, w, t) = 1$ and $\lim_{m\to\infty} N(x_m, w, t) = 0\}$). Moreover, by the necessary condition, we get

$$\forall_{t>0}\Big\{\lim_{m\to\infty} M(x_m, w, t) = \lim_{m\to\infty} M(w, x_m, t) = 1, \tag{16}$$

$$\lim_{m\to\infty} N(x_m, w, t) = \lim_{m\to\infty} N(w, x_m, t) = 0\Big\}. \tag{17}$$

Next, we calculate

$$\forall_{t>0}\forall_{m\in\mathsf{N}}\left\{M(T(w), w, t) \geq M\left(T(w), T(x_m), \frac{t}{2}\right) * M\left(T(x_m), w, \frac{t}{2}\right)\right.$$

$$
\begin{aligned}
&= M\left(T(w), T(x_m), \frac{t}{2}\right) * M\left(x_{m+1}, w, \frac{t}{2}\right) \\
&\geq \left. M\left(w, x_m, \frac{t}{2k}\right) * M\left(x_{m+1}, w, \frac{t}{2}\right)\right\},
\end{aligned}
$$

and

$$\forall_{t>0}\forall_{m\in\mathsf{N}}\left\{N(T(w), w, t) \leq N\left(T(w), T(x_m), \frac{t}{2}\right) \diamond \left(T(x_m), w, \frac{t}{2}\right)\right.$$

$$
\begin{aligned}
&= N\left(T(w), T(x_m), \frac{t}{2}\right) \diamond N\left(x_{m+1}, w, \frac{t}{2}\right) \\
&\leq \left. N\left(w, x_m, \frac{t}{2k}\right) \diamond N\left(x_{m+1}, w, \frac{t}{2}\right)\right\},
\end{aligned}
$$

It follows that,

$$\forall_{t>0}\Bigg\{M(T(w),w,t) \geq \lim_{m\to\infty} M\left(w,x_m,\frac{t}{2k}\right) * \lim_{m\to\infty} M\left(x_{m+1},w,\frac{t}{2}\right) = 1*1=1,$$

$$N(T(w),w,t) \leq \lim_{m\to\infty} N\left(w,x_m,\frac{t}{2k}\right) \diamond \lim_{m\to\infty} N\left(x_{m+1},w,\frac{t}{2}\right) = 0\diamond 0=0\Bigg\}$$

Similarly, we can calculate

$$\forall_{t>0}\forall_{m\in\mathsf{N}}\Bigg\{M(w,T(w),t) \geq M\left(w,T(w),\frac{t}{2}\right) * M\left(T(x_m),T(w),\frac{t}{2}\right)$$

$$\begin{aligned} &= M\left(w,x_{m+1},\frac{t}{2}\right)*M\left(T(x_m),T(w),\frac{t}{2}\right)\\ &\geq M\left(w,x_{m+1},\frac{t}{2}\right) * M\left(x_m,w,\frac{t}{2k}\right)\Bigg\},\end{aligned}$$

$$\forall_{t>0}\forall_{m\in\mathsf{N}}\Bigg\{N(w,T(w),t) \leq N\left(w,T(w),\frac{t}{2}\right) \diamond N\left(T(x_m),T(w),\frac{t}{2}\right)$$

$$\begin{aligned} &= N\left(w,x_{m+1},\frac{t}{2}\right)\diamond N\left(T(x_m),T(w),\frac{t}{2}\right)\\ &\leq N\left(w,x_{m+1},\frac{t}{2}\right) \diamond N\left(x_m,w,\frac{t}{2k}\right)\Bigg\},\end{aligned}$$

which gives,

$$\forall_{t>0}\Bigg\{M(w,T(w),t) \geq \lim_{m\to\infty} M\left(w,x_{m+1},\frac{t}{2}\right) * \lim_{m\to\infty} M\left(x_m,w,\frac{t}{2k}\right) = 1*1=1\Bigg\}$$

$$\forall_{t>0}\Bigg\{N(w,T(w),t) \leq \lim_{m\to\infty} N\left(w,x_{m+1},\frac{t}{2}\right) \diamond \lim_{m\to\infty} N\left(x_m,w,\frac{t}{2k}\right) = 0\diamond 0=0\Bigg\}$$

Now, we obtain $w = T(w)$, i.e., w is a fixed point of T in X.
Moreover,

$$\forall_{t>0}\Bigg\{M(w,w,t) \geq M\left(w,T(w),\frac{t}{2}\right) * M\left(T(w),w,\frac{t}{2}\right) = 1*1=1\Bigg\}$$

$$\forall_{t>0}\Bigg\{N(w,w,t) \leq N\left(w,T(w),\frac{t}{2}\right) \diamond N\left(T(w),w,\frac{t}{2}\right) = 0\diamond 0=0\Bigg\}$$

Now, if we define the sequence $(y_m = w : m \in \mathsf{N})$, then we have

$$\forall_{t>0}\forall_{p\in \mathsf{N}}\left\{\lim_{m\to\infty} M(x_m, x_{m+p}, t) = 1, \lim_{m\to\infty} N(x_m, x_{m+p}, t) = 0\right\}$$

and

$$\forall_{t>0}\left\{\lim_{m\to\infty} M(x_m, y_m, t) = 1, \lim_{m\to\infty} N(x_m, y_m, t) = 0\right\}.$$

Therefore, we have

$$\forall_{t>0}\left\{\lim_{m\to\infty} M(x_m, y_m, t) = 1, \lim_{m\to\infty} N(x_m, y_m, t) = 0\right\}$$

which gives

$$\forall_{t>0}\left\{\lim_{m\to\infty} M(x_m, w, t) = 1, \lim_{m\to\infty} N(x_m, w, t) = 0\right\}$$

Step 4 : Finally we see that w is a unique fixed point of T in X and $N(w, w, t) = 1$, for all $t > 0$. Indeed, assume that $T(v) = v$ for some $v \in X$. Then using $(B1)$ and $(B2)$ we obtain

$$\forall_{t>0}\forall_{m\in \mathsf{N}}\left\{1 \geq M(v, w, t) = M(T(v), T(w), t) \geq M\left(v, w, \frac{t}{k}\right) = M\left(T(v), T(w), \frac{t}{k}\right)\right.$$
$$\left.\geq M\left(v, w, t\right) \geq .... \geq M\left(v, w, \frac{t}{k^m}\right)\right\}$$

and

$$\forall_{t>0}\forall_{m\in \mathsf{N}}\left\{0 \leq N(v, w, t) = N(T(v), T(w), t) \leq N\left(v, w, \frac{t}{k}\right) = N\left(T(v), T(w), \frac{t}{k}\right)\right.$$
$$\left.\leq N\left(v, w, t\right) \leq .... \leq N\left(v, w, \frac{t}{k^m}\right)\right\}$$

which gives,

$$\forall_{t>0}\left\{M(v, w, t) \geq \lim_{m\to\infty} M\left(v, w, \frac{t}{k^m}\right) = 1\right\}$$
$$\forall_{t>0}\left\{N(v, w, t) \leq \lim_{m\to\infty} N\left(v, w, \frac{t}{k^m}\right) = 0\right\}$$

Similarly, we calculate

$$\forall_{t>0}\{1 \geq M(w, v, t) \geq \lim_{m\to\infty} M(w, v, \frac{t}{k^m}) = 1\}$$

and

$$\forall_{t>0}\{0 \leq N(w, v, t) \leq \lim_{m\to\infty} N(w, v, \frac{t}{k^m}) = 0\}$$

. Hence,

$$\forall_{t>0}\{M(v,w,t)=1\wedge M(w,v,t)=1\},\{N(v,w,t)=0\vee N(w,v,t)=0\}.$$

Next, applying previous Lemma, we get $v=w$, thus the fixed point of T is unique. Moreover $\forall_{t>0}\{M(w,w,t)=1, N(w,w,t)=0\}$. □

Theorem 3.2 : Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space, and let (M,N) be a GV-extended intuitionistic fuzzy metric on $X^2\times[0,\infty)$ such that (M,N)-intuitionistic fuzzy contractive sequences, i.e.,

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{m\in\mathsf{N}}\left\{\frac{1}{M(x_{m+1},x_{m+2},t)}-1\le k\left(\frac{1}{M(x_m,x_{m+1},t)}-1\right)\right\},$$

and

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{m\in\mathsf{N}}\left\{\frac{1}{N(x_{m+1},x_{m+2},t)}-1\ge k\left(\frac{1}{N(x_m,x_{m+1},t)}-1\right)\right\},$$

are N-GV-Cauchy. Let $T:X\to X$ be an N-GS-contraction of Banach type (in the sense of Gregori and Sapena), i.e., a mapping satisfying

(B2)

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{x,y\in X}\left\{\frac{1}{M(T(x),T(y),t)}-1\le k\left(\frac{1}{M(x,y,t)}-1\right)\right\}$$

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{x,y\in X}\left\{\frac{1}{N(T(x),T(y),t)}-1\ge k\left(\frac{1}{N(x,y,t)}-1\right)\right\}.$$

We assume that an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is N-GV-complete. Then T has a unique fixed point $w\in X$, and for each $x\in X$, the sequence $(x_m=T^m(x_0):x_0=x, m\in N)$, is convergent to w. Moreover, $M(w,w,t)=1, N(w,w,t)=0$, for all $t>0$.

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