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COMMON FIXED POINT THEOREMS IN CONE RECTANGULAR METRIC SPACES

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Abstract

Cone rectangular metric spaces were introduced by A. Azam, M. Arshad and I. Beg [1] and the Banach contraction principle was proved by them. M. Jleli and B. Samet [8] proved the Kannan's fixed point theorem in cone rectangular metric spaces. We extend the results to three self maps and prove the existence of common fixed points in these spaces.

1. Introduction and Preliminaries

Cone metric spaces were introduced by L. G. Huang and X. Zhang [7]. They proved fixed point theorems for contractive type mappings in a normal cone metric space.In [12], Rezapour and Hamlbrani proved results in [7] removing the condition of normality

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of the cone. Some more results in cone metric spaces were proved in [2], [3] and [4].
Following A. Branciari [5], cone rectangular metric spaces were introduced by A. Azam, M. Arshad and I. Beg [1] in which they replaced the triangular inequality in a metric by the rectangular inequality and proved the Banach contraction principle for such spaces. The Kannan's fixed point theorem was proved by M. Jleli and B. Samet [8] in cone rectangular metric spaces.
Many useful results on cone rectangular spaces have been proved in [6], [9], [10], [11] and [13]. In this paper, we have proved common fixed point theorems for three self maps which are weakly compatible. They are extensions of several known results in the literature.
Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is closed, nonempty,and $P \neq \{\theta\}$.

(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$.

(iii) $x \in Pand - x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$ we define a partial ordering $\leq$ with respect to P by:

$$x \leq y \Leftrightarrow y - x \in P$$

We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$,int P denotes the interior of P.

The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \Rightarrow \|x\| \leq k \|y\|$$

where$\|.\|$ is the norm in E.Here number k is called the normal constant of P.
In the following we always suppose that E is a Banach space, P is a cone in E with int$P \neq \phi$ and $\leq$ is partial ordering with respect to P.
Definition 1.1 [7] : Let X be a nonempty set.If the mapping $\rho : X \times X \to E$ satisfies:

(a) $\theta < \rho(x, y)$ for all $x, y \in X, x \neq y$ and $\rho(x, y) = \theta$ if and only if $x = y$.

(b) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.

(c) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, for all $x, y, z \in X$.

Then (X, ρ) is a cone metric space.

The following remark will be useful in proving the results which follow:

Remark 1.2 [10] : Let P be a cone in a real Banach space E and let $a, b, c \in P$,let P^0 denote the interior of P then,

(a)If $a \leq b$ and $b \ll c$, then $a \ll c$.

(b)If $a \ll b$ and $b \ll c$,then $a \ll c$.

(c)If $\theta \leq u \ll c$, for each $c \in P^0$, then $u = \theta$

(d)If $c \in P^0$ and $a_n \to \theta$,then there exists,$n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have $a_n \ll c$.

(e)If $\theta \leq a_n \leq b_n$, for each n and $a_n \to a, b_n \to b$, then $a \leq b$.

(f)If $a \leq \lambda a$, where $0 < \lambda < 1$, then $a = \theta$.

The concept of cone metric spaces is more general than that of metric spaces since each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty)$.

Definition 1.3 [1] : Let X be a nonempty set.If the mapping $d : X \times X \to E$ satisfies:

(a) $\theta < d(x, y)$ for all $x, y \in X, x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$.

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(c) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and for all distinct points $u, v \in X \setminus \{x, y\}$ { rectangular property }.

Here d is called a cone rectangular metric on X, and $(X, d)$is called a cone rectangular metric space.

Example 1.4 [8] : Let $X = \mathbb{R}, E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$

Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y; \\ (3a,3) & \text{if x and y are both in } \{1,2\}, x \neq y; \\ (a,1) & \text{if x and y are not both at a time in } \{1,2\}, x \neq y \end{cases}$$

where $a > 0$ is a constant.Then (X, d) is a cone rectangular metric space.

But it is not a cone metric space since $d(1, 2) = (3a, 3) > d(1, 3) + d(3, 2) = (2a, 2)$,the triangle inequality does not hold true.

Definition 1.5 [8] : Let (X, d) be a cone rectangular metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E, c \gg \theta$ there is N such that for all

$n > N, d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent to x and x is the limit of $\{x_n\}$. This is denoted be $x_n \to x$ as $n \to +\infty$.

Definition 1.6 [**8**] : Let (X, d) be a cone rectangular metric space,$\{x_n\}$ be a sequence in X. If for any $c \in X$ with $\theta \ll c$, there is N such that for all $n, m > N, d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 1.7 [**8**] : Let (X, d) be a cone rectangular metric space.If every Cauchy sequence is convergent in X, then X is called a complete cone rectangular metric space.

Definition 1.8 [**4**] : Let f and g be two self maps of a nonempty set X. If $fx = gx = y$ for some $x \in X$, then x is called the coincidence point of f and g and y is called the point of coincidence of f and g.

Definition 1.9 : Two self mappings f and g are said to be weakly compatible if they commute at their coincidence points, that is $fx = gx$ implies that $fgx = gfx$.

Proposition [**4**] : If f and g are weakly compatible self maps of a nonempty set X such that they have a unique point of coincidence i.e. $fx = gx = y$, then y is the unique common fixed point of f and g.

2. Main Results

In this section we prove two fixed point theorems for cone rectangular metric spaces.

Theorem 2.1 : Let (X, d) be a cone rectangular metric space and suppose the mappings $f, g, h : X \to X$ satisfy

$$d(fx, gy) \leq \lambda d(hx, hy) \tag{1}$$

for all $x, y \in X$ where $\lambda \in [0, 1)$. If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is a complete subspace of X, then f, g and h have a unique point of coincidence. Moreover, if (f, h) and (g, h) are weakly compatible, then f, g and h have a unique common fixed point.

Proof : Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as $hx_1 = fx_0$. This can be done since $f(X) \subseteq h(X)$. Also we can choose x_2 such that $hx_2 = gx_1$. Continuing this process having chosen x_n we can choose x_{n+1} such that
$hx_{n+1} = fx_n$ and $hx_{n+2} = gx_{n+1}, n = 0, 1, 2, \cdots$.
If $hx_n = hx_{n+1}$, then $hx_n = fx_n = gx_n$, and x_n is a coincidence point of f, g and h.

Hence assuming $x_n \neq x_{n+1}$, for $n = 0, 1, 2...$, we have

$$\begin{aligned} d(hx_n, hx_{n+1}) &= d(fx_{n-1}, gx_n) \\ &\leq \lambda d(hx_{n-1}, hx_n) \\ &\leq \lambda^2 d(hx_{n-2}, hx_{n-1}) \\ &\vdots \\ &\leq \lambda^n d(hx_0, hx_1) \end{aligned}$$

Hence,

$$d(hx_n, hx_{n+1}) \leq \lambda^n d(hx_0, hx_1) \tag{2}$$

Again,

$$\begin{aligned} d(hx_n, hx_{n+2}) &= d(fx_{n-1}, gx_{n+1}) \\ &\leq \lambda[d(hx_{n-1}, hx_{n+1})] \\ &\leq \lambda[d(hx_{n-1}, hx_n) + d(hx_n, hx_{n+2}) + d(hx_{n+2}, hx_{n+1})] \\ &\leq \lambda[\lambda^{n-1} d(hx_0, hx_1) + d(hx_n, hx_{n+2}) + \lambda^n d(hx_0, hx_1)] \end{aligned}$$

$$(1-\lambda) d(hx_n, hx_{n+2}) \leq \lambda^n d(hx_0, hx_1) + \lambda^{n+1} d(hx_0, hx_1)$$

$$d(hx_n, hx_{n+2}) \leq \frac{\lambda^n (1+\lambda)}{1-\lambda} d(hx_0, hx_1)$$

$$d(hx_n, hx_{n+2}) \leq \lambda^n \beta d(hx_0, hx_1) \tag{3}$$

where $\beta = \frac{(1+\lambda)}{1-\lambda} > 0$

For the sequence $\{hx_n\}$, we consider $d(hx_n, hx_{n+p})$ in two parts, p is even and p is odd. If p is odd ,let $p = 2m + 1$, $m \geq 1$, then by (2) and the rectangle inequality, we have,

$$\begin{aligned} d(hx_n, hx_{n+2m+1}) &\leq d(hx_n, hx_{n+1}) + d(hx_{n+1}, hx_{n+2}) + ... + d(hx_{n+2m}, hx_{n+2m+1}) \\ &\leq \lambda^n d(hx_0, hx_1) + \lambda^{n+1} d(hx_0, hx_1) + ... + \lambda^{n+2m-1} d(hx_0, hx_1) \\ &\leq \frac{\lambda^n}{1-\lambda} d(hx_0, hx_1) \end{aligned}$$

If p is even, let $p = 2m$, $m \geq 2$, then by (2),(3) and the rectangle inequality, we have,

$$\begin{aligned} d(hx_n, hx_{n+2m}) &\leq d(hx_n, hx_{n+2}) + d(hx_{n+2}, hx_{n+3}) + ... + d(hx_{n+2m-1}, hx_{n+2m}) \\ &\leq \lambda^n \beta d(hx_0, hx_1) + \lambda^{n+2} d(hx_0, hx_1) + ... + \lambda^{n+2m-1} d(hx_0, hx_1) \\ &\leq \lambda^n \beta d(hx_0, hx_1) + \frac{\lambda^n}{1-\lambda} d(hx_0, hx_1) \end{aligned}$$

As $\beta > 0$ and $\lambda \in [0, 1)$, $\lambda^n \beta \to \theta$, $\frac{\lambda^n}{1-\lambda} \to \theta$, so by (a) and (d) of Remark 1.2, for every $c \in E$ with $\theta \ll c$, there exits $n_0 \in \mathbb{N}$ such that $d(hx_n, hx_{n+p}) \ll c$ for all $n > n_0$.
Hence, $\{hx_n\}$ is a Cauchy sequence in X. Since $h(X)$ is complete subspace of X, there exists points u, v in $h(X)$ such that $\quad hx_n \to v = hu$.
Let us prove $hu = fu$.
Given $c \gg \theta$, we choose natural numbers k_1, k_2 such that

$$d(v, hx_n) \ll \frac{c}{3} \quad \forall n \geq k_1, \quad d(hx_n, hx_{n+1}) \ll \frac{c}{3} \quad \forall n \geq k_2.$$

By the rectangular property,

$$\begin{aligned} d(hu, fu) &\leq d(hu, hx_n) + d(hx_n, hx_{n+1}) + d(hx_{n+1}, fu) \\ &\leq d(v, hx_n) + d(hx_n, hx_{n+1}) + d(gx_n, fu) \\ &\leq d(v, hx_n) + d(hx_n, hx_{n+1}) + \lambda d(hu, hx_n) \\ &\leq d(v, hx_n) + d(hx_n, hx_{n+1}) + d(v, hx_n) \\ &\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c \end{aligned}$$

for all $n \geq k$ where $k = max\{k_1, k_2\}$.
Since c is arbitrary,

$$d(hu, fu) \ll \frac{c}{m}, \forall m \in \mathbb{N}$$

So, $\frac{c}{m} - d(hu, fu) \in P \quad \forall m \in \mathbb{N}$.Since $\frac{c}{m} \to \theta$ as $m \to \infty$ and P is closed,$-d(hu, fu) \in P$. Hence $d(hu, fu) \in P \cap (-P)$. Since $P \cap (-P) = \theta$, $d(hu, fu) = \theta$, hence $hu = fu$.
Similarly we can prove that $hu = gu$ which implies that v is a point of coincidence of h, f and g,
i.e. $\quad hu = gu = fu = v$.

To show that h, f and g have a unique point of coincidence, let us assume that there exists another point v^* in X such that $hu^* = gu^* = fu^* = v^*$ for some u^* in X. Now,

$$\begin{aligned} d(v, v^*) &= d(fu, gu^*) \\ &\leq \lambda d(hu, hu^*) \\ &\leq \lambda d(v, v^*) \end{aligned}$$

which implies that $v = v^*$.
Also if (f, h) and (g, h) are weakly compatible, then by Proposition 1.10, f, g and h have a unique common fixed point. □

Example 2.2 : Let $E = \mathbb{R}^2$,$P = \{(x, y) \in \mathbb{R}, x, y \geq 0\}$ and $X = \{1, 2, 3, 4\}$
Define $d : X \times X \to E$ by:
$d(x, x) = (0, 0)$
$d(1, 2) = d(2, 1) = (3, 9)$
$d(2, 3) = d(3, 2) = d(1, 3) = d(3, 1) = (1, 3)$
$d(1, 4) = d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = (4, 12)$
Then (X, d) is a cone rectangular metric space.
Define mappings f, g and $h : X \to X$ as follows:

$$f(x) = 3, \forall x \in X$$

$$g(x) = \begin{cases} 3 & \text{if } x \neq 4; \\ 1 & \text{if } x = 4; \end{cases}$$

$$h(x) = x, \forall x \in X$$

It is clear that $f(X) \cup g(X) \subseteq h(X)$. Also (f, h) and (g, h) are weakly compatible. Conditions of Theorem 2.1 hold true and 3 is the unique common fixed point of f, g and h. □

Theorem 2.3 : Let (X, d) be a cone rectangular metric space and the mappings $f, g, h : X \to X$ satisfy the condition

$$d(fx, gy) \leq \alpha[d(hx, fx) + d(hy, gy)] \tag{4}$$

where $\alpha \in [0, 1/2)$.If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$is a complete subspace of X, then f, h and g have a unique point of coincidence. Moreover, if (f, h) and (g, h) are weakly compatible, then f, g and h have a unique common fixed point.

Proof : Let $x_0 \in X$. Like in previous Theorem, we define a sequence $\{hx_n\}$ in X as, $hx_{n+1} = fx_n, hx_{n+2} = gx_{n+1}, n = 0, 1, 2...$
Assuming $x_n \neq x_{n+1}$ for $n = 0, 1, 2...$, we have

$$\begin{aligned} d(hx_n, hx_{n+1}) &= d(fx_{n-1}, gx_n) \\ &\leq \alpha[d(hx_{n-1}, fx_{n-1}) + d(hx_n, gx_n)] \\ &\leq \alpha[d(hx_{n-1}, hx_n) + d(hx_n, hx_{n+1})] \end{aligned}$$

$$\begin{aligned} d(hx_n, hx_{n+1}) &\leq \frac{\alpha}{1-\alpha} d(hx_n, hx_{n-1}) \\ &\leq r d(hx_n, hx_{n-1}) \quad where \quad r = \frac{\alpha}{1-\alpha} \in [0, 1) \\ &\vdots \\ &\leq r^n d(x_0, x_1) \end{aligned}$$

Hence,

$$d(hx_n, hx_{n+1}) \leq r^n d(x_0, x_1) \tag{5}$$

Also,

$$\begin{aligned} d(hx_n, hx_{n+2}) &= d(fx_{n-1}, gx_{n+1}) \\ &\leq \alpha[d(hx_{n-1}, fx_{n-1}) + d(hx_{n+1}, gx_{n+1}] \\ &\leq \alpha[d(hx_{n-1}, hx_n) + d(hx_{n+1}, hx_{n+2})] \\ &\leq \alpha[r^{n-1} d(hx_0, hx_1) + r^{n+1} d(hx_0, hx_1)] \\ &\leq \alpha r^{n-1}(1 + r^2) d(hx_0, hx_1) \end{aligned}$$

i.e.

$$d(hx_n, hx_{n+2}) \leq \beta r^{n-1} d(hx_0, hx_1) \tag{6}$$

where $\beta = \alpha(1 + r^2) > 0$.
For the sequence $\{hx_n\}$, we consider $d(hx_n, hx_{n+p})$ in two cases, when p is odd and when p is even.
If p is even say $p = 2m$, $m \geq 2$ then using (5) and the rectangular inequality,

$$\begin{aligned} d(hx_n, hx_{n+2m}) &\leq d(hx_n, hx_{n+2}) + d(hx_{n+2}, hx_{n+3}) + ... + d(hx_{n+2m-1}, hx_{n+2m}) \\ &\leq \beta r^{n-1} d(hx_0, hx_1) + r^{n+2} d(hx_0, hx_1) + ... + r^{n+2m-1} d(hx_0, hx_1) \\ &\leq \beta r^{n-1} d(hx_0, hx_1) + \frac{r^n}{1-r} d(hx_0, hx_1) \end{aligned}$$

If p is odd say $p = 2m+1$, $m \geq 1$ then using (5),(6) and the rectangular inequality,

$$\begin{aligned} d(hx_n, hx_{n+2m+1}) &\leq d(hx_n, hx_{n+1}) + d(hx_{n+1}, hx_{n+2}) + ... + d(hx_{n+2m}, hx_{n+2m+1}) \\ &\leq r^n d(hx_0, hx_1) + r^{n+1} d(hx_0, hx_1) + ... + r^{n+2m} d(hx_0, hx_1) \\ &\leq \frac{r^n}{1-r} d(hx_0, hx_1) \end{aligned}$$

As $0 \leq r < 1$, $\beta r^{n-1} \to \theta$, $\frac{r^n}{1-r} \to \theta$, by (a) and (d)of Remark 1.2, for every $c \in E$ with $\theta \ll c$, there exits $n_0 \in \mathbb{N}$ such that $d(hx_n, hx_{n+p}) \ll c$ for all $n > n_0$. Thus $\{hx_n\}$ is a Cauchy sequence in X.

Since $h(X)$ is a complete subspace of X,there exist points $u, v \in h(X)$ such that $hx_n \to v = hu$.

We will prove $hu = fu$. By the rectangular inequality, consider

$$\begin{aligned} d(hu, fu) &\leq d(hu, hx_n) + d(hx_n, hx_{n+1}) + d(hx_{n+1}, fu) \\ &\leq d(v, hx_n) + d(hx_n, hx_{n+1}) + d(gx_n, fu) \\ &\leq d(v, hx_n) + d(hx_n, hx_{n+1}) + \alpha[d(hu, fu) + d(hx_n, gx_n)] \\ &\leq d(v, hx_n) + d(hx_n, hx_{n+1}) + \alpha[d(hu, fu) + d(hx_n, hx_{n+1})] \end{aligned}$$

$$d(hu, fu) \leq \frac{1}{1-\alpha}[d(v, hx_n) + (1+\alpha) d(hx_n, hx_{n+1})]$$

Given $c \gg \theta$, we choose natural numbers k_3, k_4 such that

$$d(v, hx_n) \ll \frac{c(1-\alpha)}{2} \quad \forall n \geq k_3 \quad d(hx_n, hx_{n+1}) \ll \frac{(1-\alpha)}{(1+\alpha)} \frac{c}{2} \quad \forall n \geq k_4$$

Hence,

$$d(hu, fu) \ll \frac{c}{2} + \frac{c}{2} = c$$

for all $n \geq k$ where $k = max\{k_3, k_4\}$.

Since c is arbitrary,

$$d(hu, fu) \ll \frac{c}{m}, \forall m \in \mathbb{N}$$

So, $\frac{c}{m} - d(hu, fu) \in P \quad \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to \theta$ as $m \to \infty$ and P is closed,$-d(hu, fu) \in P$. Hence $d(hu, fu) \in P \cap (-P)$.Therefore $d(hu, fu) = \theta$,hence $hu = fu$.

Similarly we can prove that $hu = gu$, i.e. v is the coincidence point of h, g and f.
To show that f, g and h have a unique point of coincidence, let us assume there exists points $u^*, v^* \in X$ such that $hu^* = fu^* = gu^* = v^*$.
Now,

$$\begin{aligned} d(v, v^*) &= d(fu, gu^*) \\ &\leq \alpha[d(hu, fu) + d(hu^*, gu^*)] \\ &\leq \alpha[d(v, v) + d(v^*, v^*)] \end{aligned}$$

which implies that $v = v^*$.
Also by Proposition 1.10,if (f, h) and (g, h) are weakly compatible,then f, g and h have a unique common fixed point. □

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