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ON ϕ-RECURRENT PARA-KENMOTSU MANIFOLDS

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Abstract

The objective of the present paper is to introduce the notion of ϕ-recurrent para-Kenmotsu manifold and study its various geometric properties.

1. Introduction

On the analogy of almost-contact manifolds, in 1976 Sato [10] introduced the notion of almost para-contact manifolds. An almost contact manifold is always odd-dimensional but an almost para-contact manifold could be of even dimension as well. Takahashi [12], defined almost contact manifolds (in particular, Sasakian manifolds) equipped with an associated pseudo-Riemannian metric. In 1985, Kaneyuki et al. [5] defined the notion of almost paracontact structure on pseudo-Riemannian manifold of dimension $n(= 2m+1)$.

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Later Zamkovoy [15] showed that any almost paracontact structure admits a pseudo-Riemannain metric with signature $(n+1, n)$. The notion of para-Kenmotsu manifold was introduced by Welyczko [14]. This structure is an analogy of Kenmotsu manifold [6] in para-contact geometry. Para-Kenmotsu (briefly p-Kenmotsu) and special para-Kenmotsu (briefly sp-Kenmotsu) manifolds was studied by Sinha et al. [11], Blaga [1] and Sai Prasad et al. [9], Prakasha et al. [8] and others.

During the last five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent. As a weaker version of locally symmetry, Takahashi [13] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. Recently, De et al. introduced and studied the notion of ϕ-recurrency on a Sasakian manifold, which generalizes the notion of locally ϕ-symmetric Sasakian manifolds. De et al. [3] and Nagaraja [7] have studied this notion to Kenmotsu and trans-Sasakian manifolds, respectively.

Ricci solitons, introduced by Hamilton [4] are natural generalizations of Einstein metrics, and is defined on a Riemannian manifold (M, g). A Ricci soliton (g, V, λ) is defined on (M, g) as

$$\pounds_V g(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.1}$$

where $\pounds_V g$ denotes the Lie derivative of Riemannian metric g along a vector feild V, λ is a constant, and X, Y are arbitrary vector fields on M. A Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero, and positive respectively. In this connection we can mention the work of Blaga [1] for η-Ricci solitons on para-Kenmotsu manifolds.

The paper is organized as follows: Section 2 consist the basic definitions of para-Kenmotsu manifolds. In section 3, we introduce and study the notion of ϕ-recurrent para-Kenmotsu manifold and prove that a Ricci soliton admitting such a type of manifold is an expanding. Also, we prove that a locally ϕ-recurrent para-Kenmotsu manifold is of constant curvature -1. Finally, it is shown that, if a ϕ-recurrent para-Kenmotsu manifold has a non-zero sectional curvature, then it reduces to a locally ϕ-symmetric manifold in the sense of Takahashi.

2. Preliminaries

Let (M^n, g) be an n-dimensional smooth manifold with an almost paracontact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$-tensor field, ξ is a vector field, η is a 1-form and g is a pseudo-Riemannian metric such that

$$\phi^2(X) = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in T(M^n)$.

If an almost paracontact metric manifold satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{2.4}$$

for any $X, Y \in TM^n$, then (M^n, g) is called a almost para-Kenmotsu manifold. A normal almost para-Kenmotsu manifold is a para-Kenmotsu manifold. The para-Kenmotsu structure for 3-dimensional normal almost paracontact metric structures was introduced by Welyczko [14].

From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.5}$$

Moreover, the curvature tensor R and the Ricci tensor S satisfy

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.7}$$

$$R(\xi, X)Y = \eta(X)Y - g(X, Y)\xi, \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.9}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.10}$$

for any vector fields $X, Y, Z \in TM^n$.

3. ϕ-recurrent Para-Kenmotsu Manifolds

Analogous of consideration of ϕ-recurrent Sasakian manifold [2], we give the following definition:

Definition 3.1 : A para-Kenmotsu manifold is said to be a ϕ-recurrent manifold if there exists a nowhere vanishing unique 1-form A such that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z. \tag{3.1}$$

for all vector fields X, Y, Z, W, where A is 1-form defined by $A(X) = g(X, \rho)$ and ρ is a vector field associated with 1-form A.

In particular, if the vector fields are horizontal, then the manifold turns to locally ϕ-recurrent para-Kenmotsu manifold.

Especially, if the 1-form A in (3.1) vanishes and the vector fields are horizontal, then the manifold reduces to a locally ϕ-symmetric para-Kenmotsu manifold.

By virtue of (2.1), the equation (3.1) becomes

$$(\nabla_W R)(X,Y)Z - \eta((\nabla_W R)(X,Y)Z)\xi = A(W)R(X,Y)Z, \tag{3.2}$$

from which it follows that

$$g((\nabla_W R)(X,Y)Z, U) - \eta((\nabla_W R)(X,Y)Z)\eta(U) = A(W)g(R(X,Y)Z, U). \tag{3.3}$$

Let $\{e_i\}, i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.3) and taking summation over $i, 1 \leq i \leq n$, we get

$$(\nabla_W S)(Y,Z) - \sum_{i=1}^{n} \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) = A(W)S(Y,Z), \tag{3.4}$$

By putting $Z = \xi$ in (3.2), the second term of L.H.S. reduces to the form

$$\sum_{i=1}^{n} \left[g((\nabla_W R)(e_i, Y)\xi, \xi) g(e_i, \xi) \right],$$

which is denoted by E. In this case E vanishes. Namely we have

$$\begin{aligned} -g((\nabla_W R)(e_i, Y)\xi, \xi) = & -g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(\nabla_W e_i, Y)\xi, \xi) \\ & + g(R(e_i, \nabla_W Y)\xi, \xi) + g(R(e_i, Y)\nabla_W \xi, \xi), \end{aligned}$$

at $p \in M$. Since $\{e_i\}$ is an orthonormal basis, $\nabla_X e_i = 0$ at p. Using (2.1) and (2.7) we obtain

$$g(R(e_i, \nabla_W Y)\xi, \xi) = g(e_i, \xi)g(\nabla_W Y, \xi) - g(\nabla_W Y, \xi)g(e_i, \xi) = 0.$$

Thus we obtain

$$-g((\nabla_W R)(e_i,Y)\xi,\xi) = -g(\nabla_W R(e_i,Y)\xi,\xi) + g(R(e_i,Y)\nabla_W\xi,\xi).$$

In virtue of $g(R(e_i,Y)\xi,\xi) = -g(R(\xi,\xi)Y,e_i) = 0$, we have

$$g(\nabla_W R(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W\xi) = 0,$$

which implies

$$-g((\nabla_W R)(e_i,Y)\xi,\xi) = g(R(e_i,Y)\xi,\nabla_W\xi) + g(R(e_i,Y)\nabla_W\xi,\xi).$$

Using (2.4) and applying the skew-symmetry of R we get

$$-g((\nabla_W R)(e_i,Y)\xi,\xi) = g(R(e_i,Y)\xi,\phi^2 W) + g(R(e_i,Y)\phi^2 W,\xi) = 0,$$

Hence, we reach

$$\begin{aligned} E &= \sum_{i=1}^{n}\left[g(R(\phi^2 W,\xi)Y,e_i)g(\xi,e_i) + g(R(\xi,\phi^2 W)Y,e_i)g(\xi,e_i)\right] \\ &= g(R(\phi^2 W,\xi)Y,\xi) + g(R(\xi,\phi^2 W)Y,\xi) = 0. \end{aligned}$$

Replacing Z by ξ in (3.4) and using (2.9) we have

$$(\nabla_W S)(Y,\xi) = -(n-1)A(W)\eta(Y). \tag{3.5}$$

Now we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W\xi).$$

Using (2.4) and (2.9) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi) = (n-1)g(W,Y) - S(Y,W), \tag{3.6}$$

In view of (3.5) and (3.6) we obtain

$$S(Y,\phi^2 W) = -(n-1)\{A(W)\eta(Y) + g(W,Y) - \eta(W)\eta(Y)\}, \tag{3.7}$$

Replacing Y by ϕY in (3.7) and then using (2.1) and (2.8) we obtain

$$S(Y,W) = -(n-1)g(Y,W), \tag{3.8}$$

for all Y, W. This leads to the following:

Theorem 3.1 : A ϕ-recurrent para-Kenmotsu manifold $(M^n, g), (n > 3)$ is an Einstein manifold.

Let (M^n, g) be an n-dimension para-Kenmotsu manifold and let (g, V, λ) be a Ricci soliton in (M^n, g). Let V be pointwise collinear with ξ, i.e $V = \xi$ on M^n. Then the relation (2.2) implies

$$(\pounds_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

or

$$2S(X, Y) = -(\pounds_\xi g)(X, Y) - 2\lambda g(X, Y). \tag{3.9}$$

for any X,$Y \in \Gamma(M)$.

On a para-Kenmotsu manifold (M^n, g), from Eq.(2.5) we obtain

$$\begin{aligned}(\pounds_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi), \\ &= 2\{g(X, Y) - \eta(X)\eta(Y)\}.\end{aligned} \tag{3.10}$$

By plugging Eq.(3.10) in Eq.(3.9), we have

$$S(X, Y) = -(\lambda + 1)g(X, Y) + \eta(X)\eta(Y). \tag{3.11}$$

Let a generalized ϕ-recurrent para-Kenmotsu manifold $(M^n, g), n > 2$, admits a Ricci soliton (g, ξ, λ). Then by Eqs.(3.8) and (3.11) we get

$$(n - 2 - \lambda)g(X, Y) + \eta(X)\eta(Y) = 0. \tag{3.12}$$

Substitution of $X = \xi$, the above leads to the relation:

$$\lambda = n - 1. \tag{3.13}$$

Therefore, λ is positive for $n > 1$. Hence, by the above discussion we are able to state:

Theorem 3.2 : A Ricci soliton (g, ξ, λ) in a generalized ϕ-recurrent para-Kenmotsu manifold $(M^n, g)(n > 2)$ is an expanding.

Now from (3.2) we have

$$(\nabla_W R)(X, Y)Z = \eta((\nabla_W R)(X, Y)Z)\xi + A(W)R(X, Y)Z, \tag{3.14}$$

From (3.14) and The Bianchi identity we have

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = 0, \tag{3.15}$$

By virtue of (2.6) we obtain from (3.15)

$$\begin{aligned} &A(W)[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)] \\ &+A(X)[\eta(W)g(Y,Z) - \eta(Y)g(W,Z)] \\ &+A(Y)[\eta(X)g(W,Z) - \eta(W)g(X,Z)] = 0. \end{aligned} \tag{3.16}$$

Putting $Y = Z = e_i$ in (3.16) and taking summation over $i, 1 \leq i \leq 2n+1$, we obtain

$$\eta(W)A(X) = \eta(X)A(W), \tag{3.17}$$

for any vector field X and W. Replacing X by ξ in (3.17), it follows that

$$A(W) = \eta(\rho)\eta(W), \tag{3.18}$$

for any vector field X. Where $A(\xi) = g(\xi, \rho) = \eta(\rho)$, ρ being the vector field associated to the 1-form A.

From (3.17) and (3.18) we can state the following:

Theorem 3.3 : In a ϕ-recurrent para-Kenmotsu manifold $(M^n, g), (n > 3)$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by (3.18).

Next, in view of (2.4) and (2.5) it can be easily seen that in a para-Kenmotsu manifold the following relation holds:

$$(\nabla_W R)(X,Y)\xi = g(X,W)Y - g(Y,W)X - R(X,Y)W, \tag{3.19}$$

By virtue of (2.6), it follows from (3.19) that

$$\eta((\nabla_W R)(X,Y)\xi) = 0. \tag{3.20}$$

In view of (3.19) and (3.20), we obtain from (3.2) that

$$(\nabla_W R)(X,Y)\xi = A(W)R(X,Y)\xi, \tag{3.21}$$

from which it follows that

$$g(X,W)Y - g(Y,W)X - R(X,Y)W = \eta(\rho)\eta(W)[\eta(X)Y - \eta(Y)X]. \tag{3.22}$$

If X and Y are horizontal vector fields, then we obtain

$$g(X,W)Y - g(Y,W)X = R(X,Y)W. \tag{3.23}$$

Hence we have the following:

Theorem 3.4 : A locally ϕ-recurrent para-Kenmotsu manifold $(M^n, g), (n > 3)$, is a manifold of constant curvature -1.

Now, we suppose that a para-Kenmotsu manifold $(M^n, g), (n > 3)$, is a ϕ-recurrent. Then from (3.14) and (3.19), it follows that

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= \{g(X,W)g(Y,Z) - g(Y,W)g(X,Z) \\ &- g(R(X,Y)W,Z)\}\xi + A(W)R(X,Y)Z\end{aligned} \tag{3.24}$$

From above it follows that

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z - A(W)\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi, \tag{3.25}$$

which yields

$$\phi^2((\nabla_W R)(X,Y)Z = A(W)R(X,Y)Z,$$

provided that the vector fields X and Y are horizontal. Which states the following

Theorem 3.5 : A para-Kenmotsu manifold $(M^n, g), (n > 3)$, satisfying the relation (3.24) is locally ϕ-recurrent.

Next we suppose that in a ϕ-recurrent para-Kenmotsu manifold, the sectional curvature of a plane $\pi \in T_P M$ defined by

$$K_P(\pi) = g(R(X,Y)Y, X)$$

is a non-zero constant k, where $\{X, Y\}$ is any orthonormal basis of π. Then we have

$$g((\nabla_W R)(X,Y)Y, X) = 0, \tag{3.26}$$

By virtue of (3.26) and (3.2) we obtain

$$-g((\nabla_W R)(X,Y)Y, \xi)\eta(X) = A(W)g(R(X,Y)Y, X), \tag{3.27}$$

Since in a ϕ-recurrent para-Kenmotsu manifold, the relation (3.24) holds good, using (3.24) in (3.27) we get

$$\begin{aligned}&-\eta(X)\{g(X,W)g(Y,Y) - g(Y,W)g(Y,X) - g(R(X,Y)W,Y)\} \\ &+A(W)\{\eta(Y)g(X,Y) - \eta(X)g(Y,Y)\} = kA(W).\end{aligned} \tag{3.28}$$

Putting $W = \xi$ in (3.28) we get,

$$\eta(\rho)\{k - \eta(Y)g(X, Y) + \eta(X)g(Y, Y)\} = 0,$$

which implies that,

$$\eta(\rho) = 0.$$

Hence by (3.18) we obtain from (3.1) that

$$\phi^2((\nabla_W R)(X, Y)Z = 0.$$

This leads to the following:

Theorem 3.6 : If a ϕ-recurrent para-Kenmotsu manifold $(M^n, g), (n > 3)$, has a non-zero constant sectional curvature, then it reduces to a locally ϕ-symmetric manifold in the sense of $Takahashi$.

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