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ON THE RIESZ-THORIN INTERPOLATION THEOREM

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Abstract

In this paper we discuss and prove the Riesz-Thorin interpolation theorem in the space of tempered distributions.

1. Introduction

In this paper, we present one classical result of interpolation operator; The Riesz-Thorin Interpolation Theorem in $L^2(\mathbb{R}^n)$. The theorem allows us to show that a linear operator that is bounded on two spaces is bounded on every space in between the two. The proof

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of the Riesz-Thorin Interpolation Theorem is based on the Hadamard's Three-Lines lemma.

Lemma 1.1 (The three lines lemma) : Suppose $F(z)$ is a bounded continuous function in the strip $Rez \in [0,1]$ which is holomorphic for $Rez \in (0,1)$. If $|F(z)| \leq\leq N_0$ for $Rez = 0$ and $|F(z)| \leq N_1$ for $Rez = 1$, then $|F(z) \leq N_0^{1-x}N_1^x$ for $Rez = x; 0 < x < 1$.

Proof : We follow the proof due to ([1],[2]).

Let $G_\epsilon(z) = G(z)N_0^{z-1}N_1^{-z}e^{\epsilon z(z-1)}$. For $Rez = 0$, we have

$$\begin{aligned} |G_\epsilon(iy)| &= |G(iy)N_0^{iy-1}N_1^{-iy}e^{\epsilon iy(iy-1)}| \\ &\leq |N_0N_0^{iy-1}N_1^{\epsilon iy(iy-1)}|; \text{since} \quad F(z) \leq N_0 \quad \text{for} \quad Rez = 0 \\ &\leq |e^{iy\ln N_0}e^{-iy\ln N_1}e^{\epsilon iy(iy-1)}| \\ &= |e^{iy\ln(N_0/N_1)}e^{-\epsilon y^2}e^{-\epsilon iy}|. \end{aligned}$$

Now since $|e^{iy\ln(N_0/N_1)}| \leq 1$; $|e^{\epsilon iy}| \leq 1$ (both are on the unit circle), $|G_\epsilon(iy)| \leq e^{\epsilon y^2}| \leq 1$. For $Rez = 1$, we have

$$\begin{aligned} |G_\epsilon(1+iy)| &= |G(1+iy)N_0^{iy}N_1^{-(1+iy)}e^{\epsilon iy(1+iy)}| \\ &\leq |e^{\ln(N_0/N_1)}e^{i\epsilon}e^{-\epsilon y^2}| \\ &\leq e^{\epsilon y^2} \leq 1. \end{aligned}$$

When $0 < Rez < 1$ i.e $0 < x < 1$,

$$\begin{aligned} |G_\epsilon(x+iy)| &= |G(x+iy)N_0^{x+iy-1}N_1^{-(x+iy)}e^{\epsilon(x+iy)(iy-1)}| \\ &\leq |G(x+iy)||N_0^{x-1}N_1^{-x}e^{\epsilon x(x-1)}|e^{-\epsilon y^2} \\ &\leq Ce^{-\epsilon y^2} \end{aligned}$$

since $F(z)$ is bounded on the strip. Now since holomorphic functions attain their maximum and minimum on the boundary of any compact set; consider the compact domain $K = \{z||Imz| \leq k; 0 \leq Rez \leq 1\}$ where k is so large that $|G_\epsilon(x+iy)| \leq 1$ for all $|y| \geq k$ and $0 \leq x \leq 1$. Thus we see that $e^{-\epsilon y^2} \to 0$ as $y \to \infty$and hence $|G_\epsilon(x+iy)| \to 0$ as $Imz \to 0$. Hence from above, we have

$$\lim_{\epsilon \to 0} |G_\epsilon(x+iy)| \leq |G(x+iy)|N_0^{x-1}N_1^{-x} \leq 1$$

for Rez. Therefore, $|G(x+iy) \leq N_0^{1-x}N_1^x$. We also need the following lemma as well.

Lemma 1.2 (Lyapnov Inequality) : Let $1 \leq p_0$, $p_1 \leq \infty$ and $0 \leq \sigma \leq 1$. Define p by

$$\frac{1}{p} = \frac{1-\sigma}{p_0} + \frac{\sigma}{p_1}$$

Then $L^{p_0} \cap L^{p_1} \subset L^p$ and we have

$$|f||_p||f||_{p_0}^{1-\sigma}||f||_{p_1}^{\sigma}; \forall f \in L^{p_0} \cap L^{p_1} \tag{1}$$

Proof : We use Holder's inequality to prove inequality (). We have $||fg||_1 \leq ||f||_p||f||_q$ whenever $1/p + 1/q = 1$. We let $x = (1\sigma)p$; $y = \sigma p$; $1/z_0 = x/p_0$; $1/z_1 = y/p_1$, then $x + y = p$; $1/z_0 + 1/z_1 = 1$; $xz_0 = p_0$ and $yz_1 = p_1$. Hence using Holder's inequality, we get

$$\begin{aligned}
||f||_p^p &= (\int |f|^p dx)^{\frac{1}{p}} = ||f^p||_1 = ||f^{x+y}||_1 \\
&= ||f^x f^y|| \leq ||f^x||_{z_0}||f^y||_{z_1} \\
&= (\int |f|^{xz_0} dx)^{\frac{1}{z_0}} (\int |f|^{yz_1} dx)^{\frac{1}{z_1}} \\
&= (\int |f|^{p_0} dx)^{\frac{1-\sigma}{p_0}p} (\int |f|^{p_1} dx)^{\frac{\sigma}{p_1}p} \\
&= (||f||_{p_0}^{1-\sigma}||f||_p^{\sigma}.
\end{aligned}$$

Thus

$$||f||_p \leq ||f||_{p_0}^{1-\sigma}||f||_{p_1}^{\sigma}$$

Theorem 1.3 (Riesz-Thorin Interpolation theorem) : Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and $z \in (0,1)$. Define $1 \leq p, q \leq \infty$ by

$$\frac{1}{p} = \frac{1-z}{p_0} + \frac{z}{p_1}; \frac{1}{q} = \frac{1-z}{q_0} + \frac{z}{q_1}; z \subset \mathbb{C}.$$

If T is a linear map with

$$T : L^{p_0} \to L^{q_0}, ||T||_{L^{p_0}} \to L^{q_0} = N_0$$

$$T : L^{p_1} \to L^{q_1}, ||T||_{L^{p_1}} \to L^{q_1} = N_1.$$

Then we have $||Tf||_q \leq N_0^{1-z}N_1^z||f||_p$ for all $f \in L^{p_0} \cap L^{p_1}$. Hence T extends uniquely as a continuous map from L^p to L^q, with $||T||_{L^p} \to L^q \leq N_0^{1-z}N_1^z$. More precisely if $||Tf||_{q_0} \leq N_0||f||_p$ and $||Tf||_q \leq N_1||f||_q$ then $||Tf||_q \leq N_0^{1-z}N_1^z||f||_p$.

Proof : We are going to use the L^2 theory of derivatives to carry out the proof, for one, L^2 is a Hilbert space; secondly, the Fourier transform, which converts differentiation into multiplication by polynomials, is a unitary isomorphism on L^2. The resulting function spaces are known as (L^2)*Sobolev spaces*. Also we note that since Fourier transform maps tempered distributions into tempered distributions, we define the Sobolev norm of order s by

$$||f||_s^2 \equiv \int |\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi < \infty$$

. Now given ϕ, $\varphi \subset \delta$, $\delta-$ Schwartz class, we let

$$\begin{aligned} F(z) &= \left\langle T(1+|\xi|^2)^{-\frac{s}{2}}\phi, (1+|\xi|^2)^{\frac{t}{2}}\varphi \right\rangle \\ &= \int (T\phi)(1+|\xi|^2)^{-\frac{s}{2}}(1+|\xi|^2)^{\frac{t}{2}}\varphi d\xi \end{aligned}$$

$$s(z) = \frac{1}{p}, s_0 = \frac{1}{p_0}, t(z) = \frac{1}{q}, t_0 = \frac{1}{q_0}.$$

For $z = x + iy$ we have

$$\left(\int (1+|\xi|^2)^z|\phi|^2 d\xi\right)^{\frac{1}{2}} = \left(\int (1+|\xi|^2)^x|\phi|^2 d\xi\right)^{\frac{1}{2}} = ||\phi||_{s-x}$$

Since

$$|(1+|\xi|^2)^z| = |e^{z\ln(1+|\xi|^2)}| = |(1+|\xi|^2)^x e^{iy\ln(1+|\xi|^2)} \leq |(1+|\xi|^2)^x$$

for $|e^{iy\ln(1+|\xi|^2)}| < 1$. Since $(1+|\xi|^2)^z$ is an entire holomorphic function of z; it follows easily that $F(z)$ is an entire holomorphic function of z. For $Rez = 0$,

$$\begin{aligned} |F(z)| &\leq \left(\int |T\phi|^2(1+|\xi|^2)^{-s_0}d\xi\right)^{\frac{1}{2}}\left(\int |\varphi|^2(1+|\xi|^2)^{t_0}d\xi\right)^{\frac{1}{2}} \\ &\leq ||T(1+|\xi|^2)^{-s_0}\phi||_{t_0}||(1+|\xi|^2)^{t_0}\varphi||_{-t_0} \\ &\leq N_0||(1+|\xi|^2)^{-s_0}\phi||_{s_0}||(1+|\xi|^2)^{t_0}\varphi||_{t_0} \\ &\leq N_0||\phi||_0||\varphi||_0. \end{aligned}$$

Similarly for $Rez = 1$

$$\begin{aligned} |F(z)| &\leq \left(\int |T\phi|^2(1+|\xi|^2)^{-s_1}d\xi\right)^{\frac{1}{2}}\left(\int |\varphi|^2(1+|\xi|^2)^{t_1}d\xi\right)^{\frac{1}{2}} \\ &\leq ||T(1+|\xi|^2)^{-s_1}\phi||_{t_1}||(1+|\xi|^2)^{t_1}\varphi||_{-t_1} \\ &\leq N_1||(1+|\xi|^2)^{-s_1}\phi||_{s_1}||(1+|\xi|^2)^{t_1}\varphi||_{t_1} \\ &\leq N_1||\phi||_0||\varphi||_0. \end{aligned}$$

For $Rez = x$, $0 \leq x \leq 1$

$$\begin{aligned}
|F(z)| &\leq \left(\int |T\phi|^2(1+|\xi|^2)^{-s(z)}d\xi\right)^{\frac{1}{2}} \left(\int |\varphi|^2(1+|\xi|^2)^{t(z)}d\xi\right)^{\frac{1}{2}} \\
&\leq \left(\int |T\phi|^2(1+|\xi|^2)^{-(1-x-iy)s_0-(x-iy)s_1}d\xi\right)^{\frac{1}{2}} \left(\int |\varphi|^2)^{(1-x-ty)t_0+(x-iy)t_1}d\xi\right)^{\frac{1}{2}} \\
&\leq ||T(1+|\xi|^2)^{-(1-x)s_0-xs_1}\phi||_{t_0}||(1+|\xi|^2)^{(1-x)t_0+xt_1}\varphi||_{-t_0} \\
&\leq N_1||(1+|\xi|^2)^{-s_1}\phi||_{s_1}||(1+|\xi|^2)^{t_1}\varphi||_{t_1} \\
&\quad N_0||(1+|\xi|^2)^{-(1-x)s_0-xs_1}\phi||_{s_0}||(1+|\xi|^2)^{(1-x)t_0+xt_1}\varphi||_{t_0} \\
&\leq N_0||\phi||_{x(s_0-s_1)}||\varphi||_{x(t_1-t_0)}.
\end{aligned}$$

Thus by the three lines lemma,

$$|F(z)| \leq N_0^{1-x}N_1^x||\phi||_0||\varphi||_0, 0 \leq x \leq 1$$

Now

$$\begin{aligned}
F(z) &= \langle T(1+|\xi|^2)^{-\frac{s}{2}}\phi, (1+|\xi|^2)^{\frac{t}{2}}\varphi\rangle \\
&= \int (T\phi)(1+|\xi|^2)^{-\frac{s}{2}}(1+|\xi|^2)^{\frac{t}{2}}\varphi d\xi \\
&= \langle (1+|\xi|^2)^{\frac{t}{2}}T(1+|\xi|^2)^{-\frac{s}{2}}\phi, \varphi\rangle
\end{aligned}$$

and δ is dense in $L^2 = H = H^0$. SinceH^0 is its own dual, it means that $(1+|\xi|^2)^{\frac{t}{2}}T(1+|\xi|^2)^{-\frac{s}{2}}$ is bounded on H^0 and hence T is bounded from $H^{s(x)}$ to $H^{t(x)}$:

$$\begin{aligned}
||Tf||_{t(x)} &= \left(\int (1+|\xi|^2)^tT(1+|\xi|^2)^{-s}(1+|\xi|^2)^s f d\xi\right)^{\frac{1}{2}} \\
&= ||(1+|\xi|^2)^tT(1+|\xi|^2)^{-s}(1+|\xi|^2)^s f||_0 \\
&\leq N_0^{1-x}N_1^x||(1+|\xi|^2)^s f||_0 = N_0^{1-x}N_1^x||f||_{s(x)}
\end{aligned}$$

2. Conclusion

We note that in the space of tempered distribution, we cannot define the convolution of Fourier transform. Hence the application of Riesz-Thorin in the space of tempered distributions fails in the proof of convolution theorem, and Young's Inequality theorem for we note that convolution theorem is only defined in L^1 [3]. Proof of Young's Inequality fails because it uses convolution theorem.

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