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COMMUTATIVE MONOIDS AND MONOID HOMOMORPHISM ON LUKASIWICZ CONJUNCTION AND DISJUNCTION OPERATORS OVER INTUITIONISTIC FUZZY MATRICES

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Abstract

In this paper, in IFM we introduce two commutative monoids using the operators namely Luckasiwicz conjuntion and disjuntion operators $\oplus$ and $\odot$. In addition we discuss some properties like reflexive, symmetric, associative of the operators. Also monoid homomorphism has been defined.

1. Introduction

After the introduction of fuzzy set theory by Zadeh [8] many authors have generalized it. One ideal generalization is intuitionistic fuzzy theory by Atanassov [1,2] in which he introdued some operators and they found a promising direction of research. X. Zhang [10] studied Intuitionistic Fuzzy Value and Z. Xu [9] studied Intuitionistic Fuzzy Matrices from Intuitionistic Fuzzy value and defined the composition matrix which also

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an IFM and discuss some properties. Ronald R. Yager[11], Im et al [12], Khan S. K, Pal M and Amiya K. Shyamal [3], Meenakshi A. R and Gandhimathi [4], Sriram and Murugadas [6] and several authors studied Intuitionistic Fuzzy Matrices. In [14] Wang et.al developed a new approach to constructing an intuitionistic fuzzy similarity matrix based on IFM. In [13] A. K. Adak et.al introduced and studied some algebraic properties of generalised IFMs. On Lukasiwicz Intuitionistic Fuzzy conjunction and disjunction [5], Atanassov. K introduced some operators for IFSs. We extended the above to IFM and some results are proved in New operators for IFMs [7]. In this paper we explore some more algebraic results of the said operators and discuss the relation between other operators with these.
Finally Using the algebraic properties we conclude there exists two commutative monoids, monoid homomorphism over IFMs.

2. Preliminaries

We recollect some relevant basic definitions and results will be used later.

Definition 2.1 [1] : Let a set $X = \{x_1, x_2, ..., x_n\}$ be fixed, then an IFS can be defined as $A = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle \,|x_i \in X\}$ which assigns to each element x_i a membership degree $\mu_A(x_i)$ and a nonmembership degree $\nu_A(x_i)$, with the condition $0 \leq \mu_A(x_i)+\nu_A(x_i) \leq 1$ for every $x_i \in X$.

Definition 2.2 [9] [10] : The 2 tuple$(\mu(x_i), \nu(x_i)) = (x, x')$ called an Intutionistic Fuzzy Value such that $x + x' \leq 1$ and $x, x' \in [0, 1]$.

Definition 2.3 [1] : For $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$, define

$$\left\langle x, x'\right\rangle \vee \left\langle y, y'\right\rangle = \left\langle max\{x, y\}, min\{x', y'\}\right\rangle$$

$$\left\langle x, x'\right\rangle \wedge \left\langle y, y'\right\rangle = \left\langle min\left\{x, y\right\}, max\left\{x', y'\right\}\right\rangle$$

$$\left\langle x, x'\right\rangle^c = \left\langle x', x\right\rangle \quad \text{and} \quad \left\langle x, x'\right\rangle \ominus \left\langle y, y'\right\rangle = \begin{cases} \langle x, x'\rangle & \text{if} \quad \langle x, x'\rangle \geq \langle y, y'\rangle \\ \\ \langle 0, 1\rangle & \text{if} \quad \langle x, x'\rangle \leq \langle y, y'\rangle \end{cases}$$

Here $\langle x, x'\rangle \geq \langle y, y'\rangle$ means $x \geq y$ and $x^{'} \leq y^{'}$. And $\langle x, x'\rangle \leq \langle y, y'\rangle$ means either $x \leq y$ or $x^{'} > y^{'}$.

Definition 2.4 [9] : Let $A = (a_{ij})_{n\times n}$ be a matrix if all its elements are IFVs, then A is called an Intuitionistic Fuzzy Matrix.

Definition 2.5 [2] : An IFM $J = (\langle 1, 0\rangle)$ for all entries is known as the universal matrix and an IFM $O = (\langle 0, 1\rangle)$ for all entries is known as zero matrix. Denote the set of all IFMs of order $m \times n$ by $\mathscr{F}_{mn}$ and square matrix of order n by $\mathscr{F}_n$. The identity IFM $I = (\langle \delta_{ij}, \delta'_{ij}\rangle)$ is defined by $\langle \delta_{ij}, \delta'_{ij}\rangle = \langle 1, 0\rangle$ if $i = j$ and $\langle \delta_{ij}, \delta'_{ij}\rangle = \langle 0, 1\rangle$ if $i \neq j$.
Definition 2.6 [9] [10] : For $IFMsA = (\left\langle a_{ij}, a'_{ij}\right\rangle)_{m\times n}, B = (\left\langle b_{ij}, b'_{ij}\right\rangle)_{m\times n}$, define

$$\begin{aligned} A \vee B &= (\left\langle a_{ij}, a'_{ij}\right\rangle \vee \left\langle b_{ij}, b'_{ij}\right\rangle) \\ A \wedge B &= (\left\langle a_{ij}, a'_{ij}\right\rangle \wedge \left\langle b_{ij}, b'_{ij}\right\rangle) \\ A \ominus B &= (\left\langle a_{ij}, a'_{ij}\right\rangle \ominus \left\langle b_{ij}, b'_{ij}\right\rangle) \end{aligned}$$

Definition 2.7 [9] : For $A \in \mathscr{F}_n$

(i) A is reflexive if and only if $A \geqslant I_n$

(ii)A is symmetric if and only if $\langle a_{ij}, a'_{ij}\rangle = \langle a_{ji}, a'_{ji}\rangle$ for all i, j.

(iii)A is transitive if and only if $A \geqslant A^2$

(iv)A is irreflexive if $\langle a_{ii}, a'_{ii}\rangle = \langle 0, 1\rangle$ for alli, j.

Definition 2.8 [7] : Consider that two elements in IFS (x, x') and (y, y') such that $0 \leq x + x' \leq 1$ and $0 \leq y + y' \leq 1$

$$[(x, x') \oplus (y, y')] = [(x + y) \wedge 1, (x' + y' - 1) \vee 0]$$

$$[(x, x') \odot (y, y')] = [(x + y - 1) \vee 0, (x' + y') \wedge 1]$$

Definition 2.9 [7] : An Intuitionistic fuzzy matrix A is $A = [(a_{ij}, a'_{ij})]$ where $0 \leq a_{ij} + a'_{ij} \leq 1$ for all i, j. Obviously, every fuzzy matrix $A = [a_{ij}]$ is an intuitionistic fuzzy matrix of the form $[(a_{ij}, 1 - a_{ij})]$.
Now we define the operations $\oplus$ and $\odot$ on IFMs. Let $A = [(a_{ij}, a'_{ij})]$ and $B = [(b_{ij}, b'_{ij})]$ be two IFMs of order $m \times n$. Then

$$\begin{aligned} A \oplus B &= [((a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1) \vee 0)] \\ A \odot B &= [((a_{ij} + b_{ij} - 1) \vee 0, (a'_{ij} + b'_{ij}) \wedge 1)] \end{aligned}$$

Using these operators for any IFM $A \in F_{mn}$,the authors have proved under $\oplus$, A is compact,under $\odot$,A is transitive, $\wedge$ is distributive over $\oplus$ and $\oplus$ and $\odot$ satisfies demorgans law and also commutative.

Definition 2.10 [2] : Let M be a fixed set. Let $e_* \in M$ be an unitary element of M and let $*$ be an operation. Then $(M, *, e_*)$ is a commutative monoid if

(i) For all $a, b \in M$ implies $a * b \in M$

(ii)For all $a, b, c \in M$ implies $(a * b) * c = a * (b * c)$

(iii)For all $a \in M, a * e_* = a = e_* * a$

(iv) For all $a, b \in M, (a * b) = (b * a)$

Definition 2.11 : A homomorphism between two monoids $(M, *)$ and $(N, .)$ is a function f from M to N such that $(i) f(x * y) = f(x).f(y)$ for all$x, y \in M$
$(ii) f(e_m) = e_n$ where e_m and e_n are the identities.

3. Some Results

Proposition 3.1 : Let A and B are any two IFMs, then the following are true.

(i) If A and B are reflexive then $A \oplus B$ and $A \odot B$ are reflexive.

(ii)If A and B are irreflexive then $A \oplus B$ and $A \odot B$ are irreflexive.

(iii)If A and B are symmetric then $A \oplus B$ and $A \odot B$ are symmetric.

(iv)If A and B are nearly irreflexive then $A \oplus B$ and $A \odot B$ are weakly irreflexive.

Proof : From the defn 2.9 (i)and (ii)follows.
(iii) since A and B are symmetric $\langle a_{ij}, a'_{ij} \rangle = \langle a_{ji}, a'_{ji} \rangle$ and $\langle b_{ij}, b'_{ij} \rangle = \langle b_{ji}, b'_{ji} \rangle$.
Now let $\langle c_{ij}, c'_{ij} \rangle$ and $\langle d_{ij}, d'_{ij} \rangle$ be the ij^{th} elements of $(A \oplus B)$ and $A \odot B$.

$$\begin{aligned} \langle c_{ij}, c'_{ij} \rangle &= \langle (a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1) \vee 0 \rangle \\ &= \langle (a_{ji} + b_{ji}) \wedge 1, (a'_{ji} + b'_{ji} - 1) \vee 0 \rangle \\ &= \langle c_{ji}, c'_{ji} \rangle. \end{aligned}$$

Hence $A \oplus B$ is symmetric.
Similarly we can prove $(\langle d_{ij}, d'_{ij} \rangle = \langle d_{ji}, d'_{ji} \rangle)$. $A \odot B$ is symmetric.
(iv)Since A and B are nearly irreflexive $\langle a_{ii}, a'_{ii} \rangle \leq \langle a_{ij}, a'_{ij} \rangle$ and $\langle b_{ii}, b'_{ii} \rangle \leq \langle b_{ij}, b'_{ij} \rangle, \forall i, j$.

Let the ij^{th} element of $A \oplus B$ is

$$\langle c_{ij}, c'_{ij} \rangle = \langle (a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1) \vee 0 \rangle$$
$$\langle c_{ii}, c'_{ii} \rangle = \langle (a_{ii} + b_{ii}) \wedge 1, (a'_{ii} + b'_{ii} - 1) \vee 0 \rangle$$
$$a_{ii} \leq a_{ij} \text{ and } b_{ii} \leq b_{ij} \Rightarrow (a_{ii} + b_{ii}) \leq (b_{ij} + a_{ij})$$
$$\Rightarrow (a_{ii} + b_{ii}) \wedge 1 \leq (a_{ij} + b_{ij}) \wedge 1 \tag{2.1}$$

Similarly

$$(a'_{ii} + b'_{ii} - 1) \vee 0 > (a'_{ij} + b'_{ij} - 1) \vee 0 \tag{2.2}$$

From (2.1) and (2.2) $\langle c_{ij}, c'_{ij} \rangle \geq \langle c_{ii}, c'_{ii} \rangle$.
$A \oplus B$ is nearly irreflexive.
$A \odot B$ is nearly irreflexive can be proved analogously. □

Proposition 3.2 : For any $n \times n$ IFM A

(i) $In \oplus (A \oplus A')$ is reflexive and symmetric.

(ii) $In \oplus (A \oplus A') = In \vee (A \oplus A')$. where In is the identity IFM of order n.

Proof :

(i) Let $A = (a_{ij}, a'_{ij})$ and $A' = (a_{ji}, a'_{ji})$

$A \oplus A' = [(a_{ij} + a_{ji}) \wedge 1, (a'_{ij} + a'_{ji} - 1) \vee 0] = (s_{ij}, s'_{ij}.)$

Let $R = (r_{ij}, r'_{ij}) = In \oplus (A \oplus A')$

$$= \begin{cases} [(1 + s_{ij}) \wedge 1, (0 + s'_{ii} - 1) \vee 0] & \text{if } (i = j) \\ [(0 + s_{ij}) \wedge 1, (1 + s'_{ii} - 1) \vee 0] & \text{if } (i \neq j) \end{cases}$$

Case 1 : If $i = j$ then $(r_{ii}, r'_{ii}) = [1, 0]$ gives R is reflexive.

Case 2 : If $i \neq j$ then

$$\begin{aligned}(r_{ij}, r'_{ij}) &= [(0 + s_{ij}) \wedge 1, (1 + s'_{ij} - 1) \vee 0] = (s_{ij}, s'_{ij}) \\ &= [(a_{ij} + a_{ji}) \wedge 1, (a'_{ij} + a'_{ji} - 1) \vee 0] \\ &= [(a_{ji} + a_{ij}) \wedge 1, (a'_{ji} + a'_{ij} - 1) \vee 0] \\ &= (s_{ji}, s'_{ji}) = (r_{ji}, r'_{ji}) \Rightarrow R \text{ is symmetric.}\end{aligned}$$

(ii) If $i = j$ then $(r_{ii}, r'_{ii}) = (1, 0)$ or $(r_{ij}, r'_{ij}) = (s_{ij}, s'_{ij})$

$$\text{Let } In \vee (A \oplus A') = \begin{cases} (1, 0) \vee (s_{ij}, s'_{ij}) = (1, 0) & \text{if } i = j \\ (0, 1) \vee (s_{ij}, s'_{ij}) = (s_{ij}, s'_{ij}) & \text{if } i \neq j \end{cases}$$

$\therefore In \oplus (A \oplus A') = In \vee (A \oplus A')$. □

Proposition 3.3 : $A \vee (B \oplus C) \leq (A \vee B) \oplus (A \vee C)$.

Proof : Let $(d_{ij}, d'_{ij}) = D, (e_{ij}, e'_{ij}) = E, (f_{ij}, f'_{ij}) = F, (g_{ij}, g'_{ij}) = G$ and $(h_{ij}, h'_{ij}) = H$ are the IFMs of $B \oplus C, A \vee B, A \vee C, A \vee (B \oplus C)$ and $(A \vee B) \oplus (A \vee C)$.

Now consider the ij^{th} element as follows

$$\begin{aligned} (d_{ij}, d'_{ij}) &= [(b_{ij} + c_{ij}) \wedge 1, (b'_{ij} + c'_{ij} - 1 \vee 0)] \\ (e_{ij}, e'_{ij}) &= [(a_{ij} \vee b_{ij}), (a'_{ij} \wedge b'_{ij})] \\ (f_{ij}, f'_{ij}) &= [(a_{ij} \vee c_{ij}), (a'_{ij} \wedge c'_{ij})] \\ (g_{ij}, g'_{ij}) &= [(a_{ij} \vee d_{ij}), (a'_{ij} \wedge d'_{ij})] \\ (h_{ij}, h'_{ij}) &= [(e_{ij} + f_{ij}) \wedge 1, (e'_{ij} + f'_{ij} - 1 \vee 0)]. \end{aligned}$$

It is enough to prove that $(g_{ij}, g'_{ij}) \leq (h_{ij}, h'_{ij})$

i.e $g_{ij} \leq h_{ij}$ and $g'_{ij} > h'_{ij}$.

$$(g_{ij}, g'_{ij}) = \begin{cases} (a_{ij}, a'_{ij}) & \text{if } a_{ij} > d_{ij} \text{ and } a'_{ij} \leq d'_{ij} \\ (d_{ij}, d'_{ij}) & \text{if } d_{ij} > a_{ij} \text{ and } d'_{ij} \leq a'_{ij} \\ (a_{ij}, d'_{ij}) & \text{if } a_{ij} > d_{ij} \text{ and } d'_{ij} \leq a'_{ij} \\ (d_{ij}, a'_{ij}) & \text{if } d_{ij} > a_{ij} \text{ and } a'_{ij} \leq d'_{ij} \end{cases}$$

Case 1 : If $(a_{ij}, a'_{ij}) > (b_{ij}, b'_{ij})$ and $(a_{ij}, a'_{ij}) > (c_{ij}, c'_{ij})$ then $(e_{ij}, e'_{ij}) = (a_{ij}, a'_{ij}), (f_{ij}, f'_{ij}) = (a_{ij}, a'_{ij})$.

$$(h_{ij}, h'_{ij}) = [(2a_{ij}) \wedge 1, (2a'_{ij} - 1) \vee 0].$$

Sub case 1.1 : If $(g_{ij}, g'_{ij}) = (a_{ij}, a'_{ij}) \leq (h_{ij}, h'_{ij})$.

Sub case 1.2 : If $(g_{ij}, g'_{ij}) = (d_{ij}, d'_{ij})$ we have

$$2a_{ij} > (b_{ij} + c_{ij}) \Rightarrow (2a_{ij}) \wedge 1 > (b_{ij} + c_{ij}) \wedge 1$$

$$\Rightarrow h_{ij} > d_{ij} = g_{ij}.$$

Similarly

$$(2a'_{ij} - 1) \vee 0 \leq (b'_{ij} + c'_{ij} - 1) \vee 0 \Rightarrow h'_{ij} \leq d_{ij} = g'_{ij}.$$

$$(g_{ij}, g'_{ij}) \leq (h_{ij}, h'_{ij}).$$

Sub case 1.3 : If $(g_{ij}, g'_{ij}) = (a_{ij}, d'_{ij})$, then it is true from 1.1 and 1.2.

Case 2 : If $(a_{ij}, a'_{ij}) < (b_{ij}, b'_{ij})$ and $(a_{ij}, a'_{ij}) < (c_{ij}, c'_{ij})$ then

$$(e_{ij}, e'_{ij}) = (b_{ij}, b'_{ij}), (f_{ij}, f'_{ij}) = (c_{ij}, c'_{ij}).$$

$$\Rightarrow (h_{ij}, h'_{ij}) = [(b_{ij} + c_{ij}) \wedge 1, (b'_{ij} + c'_{ij} - 1) \vee 0] = (d_{ij}, d'_{ij}).$$

Sub case 2.1 : If $(g_{ij}, g'_{ij}) = (a_{ij}, a'_{ij})$

$$g_{ij} = a_{ij} < (b_{ij} + c_{ij}) \wedge 1 = h_{ij} \Rightarrow g_{ij} < h_{ij}.$$

Similarly $a'_{ij} \geq h'_{ij}$

$$(g_{ij}, g'_{ij}) \leq (h_{ij}, h'_{ij}).$$

Sub case 2.2 : If $(g_{ij}, g'_{ij}) = (d_{ij}, d'_{ij}) = (h_{ij}, h'_{ij})$.

Sub case 2.3 and 2.4 obvious from 2.1 and 2.2.

Case 3 : If $(a_{ij}, a'_{ij}) < (b_{ij}, b'_{ij})$ and $(a_{ij}, a'_{ij}) > (c_{ij}, c'_{ij})$ then $(h_{ij}, h'_{ij}) = [(b_{ij} + a_{ij}) \wedge 1, (b'_{ij} + a'_{ij} - 1) \vee 0]$.

Sub case 3.1 : If $(g_{ij}, g'_{ij}) = (a_{ij}, a'_{ij})$

$$(a_{ij} + b_{ij}) \wedge 1 > a_{ij} \Rightarrow h_{ij} > g_{ij} \text{ and}$$

$$b'_{ij} - 1 \leq 0 \Rightarrow (a'_{ij} + b'_{ij} - 1) \leq a'_{ij}$$

$$\Rightarrow (a'_{ij} + b'_{ij} - 1) \vee 0 \leq a'_{ij} \Rightarrow h'_{ij} \leq g'_{ij}.$$

$$(g_{ij}, g'_{ij}) \leq (h_{ij}, h'_{ij}).$$

Sub case 3.2 : If $(g_{ij}, g'_{ij}) = (d_{ij}, d'_{ij})$

$$a_{ij} > c_{ij} \Rightarrow (a_{ij} + b_{ij}) \wedge 1 > (b_{ij} + c_{ij}) \wedge 1 \Rightarrow h_{ij} > d_{ij} = g_{ij}.$$

Similarly

$$h'_{ij} \leq d'_{ij} = g'_{ij} \Rightarrow (g_{ij}, g'_{ij}) \leq (h_{ij}, h'_{ij}).$$

Sub case 3.3 and 3.4 obvious from 3.1 and 3.2.

Case 4 : If $(a_{ij}, a'_{ij}) > (b_{ij}, b'_{ij})$ and $(a_{ij}, a'_{ij}) < (c_{ij}, c'_{ij})$ then $(h_{ij}, h'_{ij}) = [(a_{ij} + c_{ij}) \wedge 1, (a'_{ij} + c'_{ij} - 1) \vee 0]$.

The proof of 4 sub cases are similarly to case 3. Using the above 4 cases $G \leq H$. □

Proposition 3.4 : (i)$A \oplus (B \ominus C) \geq (A \oplus B) \ominus (A \oplus C)$.

(ii)$A \ominus (B \oplus C) \leq (A \ominus B) \oplus (A \ominus C)$.

Proof :

(i) Let the ij^{th} element of $B \ominus C, A \oplus B, A \oplus C, A \oplus (B \ominus C)$ and $(A \oplus B) \ominus (A \oplus C)$ are as follows:

$$(d_{ij}, d'_{ij}) = \begin{cases} (b_{ij}, b'_{ij}) & \text{if } (b_{ij}, b'_{ij}) \geq (c_{ij}, c'_{ij}) \\ (0,1) & \text{otherwise} \end{cases}$$

$$(e_{ij}, e'_{ij}) = [(a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1) \vee 0]$$

$$(f_{ij}, f'_{ij}) = [(a_{ij} + c_{ij}) \wedge 1, (a'_{ij} + c'_{ij} - 1) \vee 0]$$

$$(g_{ij}, g'_{ij}) = [(a_{ij} + d_{ij}) \wedge 1, (a'_{ij} + d'_{ij} - 1) \vee 0]$$

$$(h_{ij}, h'_{ij}) = \begin{cases} (e_{ij}, e'_{ij}) & \text{if } (e_{ij}, e'_{ij}) \geq (f_{ij}, f'_{ij}) \\ (0,1) & \text{otherwise} \end{cases}$$

$$(g_{ij}, g'_{ij}) = \begin{cases} (a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1) \vee 0 & \text{if } (b_{ij}, b'_{ij}) \geq (c_{ij}, c'_{ij}) \\ (a_{ij}, a'_{ij}) & \text{if } (b_{ij}, b'_{ij}) < (c_{ij}, c'_{ij}) \end{cases}$$

It is enough to prove that $(g_{ij}, g'_{ij}) \geq (h_{ij}, h'_{ij})$.

Case 1 : If $(e_{ij}, e'_{ij}) \geq (f_{ij}, f'_{ij})$ then $b_{ij} \geq c_{ij}$ and $b'_{ij} < c'_{ij} \Rightarrow (g_{ij}, g'_{ij}) = (e_{ij}, e'_{ij}) = (h_{ij}, h'_{ij})$.

Case 2 : If $(e_{ij}, e'_{ij}) < (f_{ij}, f'_{ij})$ then $(b_{ij}, b'_{ij}) \leq (c_{ij}, c'_{ij})$. Now $(g_{ij}, g'_{ij}) = (a_{ij}, a'_{ij})$ and $(h_{ij}, h'_{ij}) = (0,1) < (g_{ij}, g'_{ij})$.

(ii) Let the ij^{th} element of $B \oplus C, A \ominus B, A \ominus C, A \ominus (B \oplus C)$ and $(A \ominus B) \oplus (A \ominus C)$ are as follows:

$$(d_{ij}, d'_{ij}) = [(b_{ij} + c_{ij}) \wedge 1, (b'_{ij} + c'_{ij} - 1) \vee 0]$$

$$(e_{ij}, e'_{ij}) = \begin{cases} (a_{ij}, a'_{ij}) & \text{if } (a_{ij}, a'_{ij}) \geq (b_{ij}, b'_{ij}) \\ (0,1) & \text{if } (a_{ij}, a'_{ij}) < (b_{ij}, b'_{ij}) \end{cases}$$

$$(f_{ij}, f'_{ij}) = \begin{cases} (a_{ij}, a'_{ij}) & \text{if } (a_{ij}, a'_{ij}) \geq (c_{ij}, c'_{ij}) \\ (0,1) & \text{if } (a_{ij}, a'_{ij}) < (c_{ij}, c'_{ij}) \end{cases}$$

$$(g_{ij}, g'_{ij}) = \begin{cases} (a_{ij}, a'_{ij}) & \text{if } (a_{ij}, a'_{ij}) \geq (d_{ij}, d'_{ij}) \\ (0,1) & \text{if } (a_{ij}, a'_{ij}) < (d_{ij}, d'_{ij}) \end{cases}$$

$$(h_{ij}, h'_{ij}) = [(e_{ij} + f_{ij}) \wedge 1, (e'_{ij} + f'_{ij} - 1) \vee 0]$$

Case 1 : If $(a_{ij}, a'_{ij}) \geq (b_{ij}, b'_{ij})$ and $(a_{ij}, a'_{ij}) \geq (c_{ij}, c'_{ij})$ then $(h_{ij}, h'_{ij}) = [(2a_{ij}) \wedge 1, (2a'_{ij} - 1) \vee 0]$.

Sub case 1.1 : If $(a_{ij}, a'_{ij}) \geq (d_{ij}, d'_{ij})$ then $(g_{ij}, g'_{ij}) = (a_{ij}, a'_{ij}) \leq (h_{ij}, h'_{ij})$.

Sub case 1.2 : If $(a_{ij}, a'_{ij}) < (d_{ij}, d'_{ij})$ Proof is obvious. Similarly we can prove the other 3 cases. □

Proposition 3.5 : (i)$(A \oplus B)^c \leq A^c \oplus B^c$

(ii)$(A \odot B)^c \geq A^c \odot B^c$.

Proposition 3.6 : The operators $\oplus$ and $\odot$ are associative.

(i) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

(ii) $(A \odot B) \odot C = A \odot (B \odot C)$ for any three IFMS A, B and C.

Proof :

(i) Let the ij^{th} elements of the IFMs $A \oplus B, B \oplus C, (A \oplus B) \oplus C$ and $A \oplus (B \oplus C)$ are as follows:

$$\begin{aligned}
(d_{ij}, d'_{ij}) &= [(a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1) \vee 0] \\
(e_{ij}, e'_{ij}) &= [(b_{ij} + c_{ij}) \wedge 1, (b'_{ij} + c'_{ij} - 1) \vee 0] \\
(f_{ij}, f'_{ij}) &= [(d_{ij} + c_{ij}) \wedge 1, (d'_{ij} + c'_{ij} - 1) \vee 0] \\
(g_{ij}, g'_{ij}) &= [(a_{ij} + e_{ij}) \wedge 1, (a'_{ij} + e'_{ij} - 1) \vee 0]
\end{aligned}$$

Now we have to prove $(f_{ij}, f'_{ij}) = (g_{ij}, g'_{ij})$.

Case 1 : If $a_{ij}+b_{ij} < 1$ and $a'_{ij}+b'_{ij}-1 \leq 0$ then $(f_{ij}, f'_{ij}) = [(a_{ij}+b_{ij}+c_{ij}) \wedge 1, 0]$.

Sub case 1.1 : If $a_{ij} + b_{ij} + c_{ij} < 1$ then $f_{ij} = (a_{ij} + b_{ij} + c_{ij},)$ in this case $b_{ij}+c_{ij} \leq 1$ then $g_{ij} = (a_{ij}+b_{ij}+c_{ij}) \wedge 1 = a_{ij}+b_{ij}+c_{ij} = f_{ij}$. Since $a'_{ij}+b'_{ij}-1 \leq 0$, either $b'_{ij} + c'_{ij} - 1 \leq 0$ or > 0.
Suppose $b'_{ij} + c'_{ij} - 1 \leq 0$ then $g'_{ij} = 0 = f'_{ij}$.
Suppose $b'_{ij} + c'_{ij} - 1 > 0$ then $g'_{ij} = [(a'_{ij} + b'_{ij} + c'_{ij} - 1 - 1) \vee 0] = [\{(a'_{ij} + b'_{ij} - 1) + (c'_{ij} - 1)\} \vee 0] = 0 = f'_{ij}$.

Sub case 1.2 : If $a_{ij}+b_{ij}+c_{ij} > 1$ then $(f_{ij}, f'_{ij}) = (1, 0)$. In this case $b_{ij}+c_{ij} \leq 1$ or > 1. If $b_{ij} + c_{ij} \leq 1, (g_{ij}, g'_{ij}) = [(a_{ij} + e_{ij}) \wedge 1, 0] = [(a_{ij} + b_{ij} + c_{ij} \wedge 1, 0)] = [1, 0] = (f_{ij}, f'_{ij})$. If $b_{ij} + c_{ij} > 1, (g_{ij}, g'_{ij}) = [(a_{ij} + 1) \wedge 1, 0] = [1, 0] = (f_{ij}, f'_{ij})$.

Case 2 : If $a_{ij} + b_{ij} \geq 1$ and $a'_{ij} + b'_{ij} - 1 < 0$ then $(f_{ij}, f'_{ij}) = (1, 0)$. Since $a_{ij} + b_{ij} \geq 1 \Rightarrow a_{ij} + b_{ij} + c_{ij} \geq 1$ and $b'_{ij} + c'_{ij} - 1 \leq 0$ or ≥ 0.
$\Rightarrow b_{ij} + c_{ij} \geq 1$ or ≤ 1 and $b'_{ij} + c'_{ij} - 1 \leq 0$ or ≥ 0.

Sub case 2.1 : If $b_{ij}+c_{ij} \geq 1$ and $b'_{ij}+c'_{ij}-1 \leq 0$ then $(g_{ij}, g'_{ij}) = (1, 0) = (f_{ij}, f'_{ij})$.

Sub case 2.2 : If $b_{ij} + c_{ij} \geq 1$ and $b'_{ij} + c'_{ij} - 1 \geq 0$
$(g_{ij}, g'_{ij}) = [1, (a'_{ij} + b'_{ij} + c'_{ij} - 1 - 1) \vee 0] = [1, \{(a'_{ij} + b'_{ij} - 1) + (c'_{ij} - 1) \vee 0] = [1, 0] = (f_{ij}, f'_{ij})$.

Sub case 2.3 : If $b_{ij} + c_{ij} < 1$ and $b'_{ij} + c'_{ij} - 1 < 0$ then $(g_{ij}, g'_{ij}) = [(a_{ij} + b_{ij} + c_{ij}) \wedge 1, (a'_{ij} + 0 - 1) \vee 0] = [1, 0] = (f_{ij}, f'_{ij})$.

Sub case 2.4 : If $b_{ij} + c_{ij} < 1$ and $(b'_{ij} + c'_{ij} - 1) > 0$.
This is true from 2.2 and 2.3.

Case 3 : If $a_{ij} + b_{ij} \leq 1$ and $a'_{ij} + b'_{ij} - 1 > 0$ then
$(f_{ij}, f'_{ij}) = [(a_{ij} + b_{ij} + c_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1 + c'_{ij} - 1) \vee 0]$.

Sub case 3.1 : If $a_{ij} + b_{ij} + c_{ij} < 1$ and $a'_{ij} + b'_{ij} + c'_{ij} - 2 < 0$.
Using sub case 1.1 $f_{ij} = g_{ij}$ and $f'_{ij} = 0$.
Now $a'_{ij} + b'_{ij} + c'_{ij} - 2 = (a'_{ij} - 1) + (b'_{ij} + c'_{ij} - 1) < 0$.
Either $b'_{ij} + c'_{ij} - 1 < 0$ or > 0.
If $b'_{ij} + c'_{ij} - 1 < 0$ then $g'_{ij} = (a'_{ij} + 0 - 1) \vee 0 = 0 = f'_{ij}$.
If $b'_{ij} + c'_{ij} - 1 \geq 0$ then $g'_{ij} = (a'_{ij} + b'_{ij} + c'_{ij} - 1 - 1) \vee 0 = 0 = f'_{ij}$.

Sub case 3.2 : If $a_{ij}+b_{ij}+c_{ij} < 1$ and $a'_{ij}+b'_{ij}+c'_{ij}-2 > 0$ then $f_{ij} = a_{ij}+b_{ij}+c_{ij}$ from sub case 1.1 $g_{ij} = f_{ij}$
$f'_{ij} = a'_{ij} + b'_{ij} + c'_{ij} - 2 = (a'_{ij} - 1) + (b'_{ij} + c'_{ij} - 1) \geq 0 \Rightarrow b'_{ij} + c'_{ij} - 1 > 0$
$\Rightarrow g'_{ij} = a'_{ij} + b'_{ij} + c'_{ij} - 2 = f'_{ij}$.

Sub case 3.3 : If $a_{ij} + b_{ij} + c_{ij} > 1$ and $a'_{ij} + b'_{ij} + c'_{ij} - 2 < 0$ from 1.3 and 3.1 this is true.

Sub case 3.4 : If $a_{ij} + b_{ij} + c_{ij} > 1$ and $a'_{ij} + b'_{ij} + c'_{ij} - 2 > 0$ then from 1.3 and 3.2 $(f_{ij}, f'_{ij}) = (g_{ij}, g'_{ij})$.

Case 4 : If $a_{ij} + b_{ij} \geq 1$ and $a'_{ij} + b'_{ij} - 1 \geq 0$ then from case 2 and 3 it is obvious.

(ii) $(A \odot B) \odot C = A \odot (B \odot C)$.
Let $(d_{ij}, d'_{ij}) = [(a_{ij} + b_{ij} - 1) \vee 0, (a'_{ij} + b'_{ij}) \wedge 1]$
$(e_{ij}, e'_{ij}) = [(b_{ij} + c_{ij} - 1) \vee 0, (b'_{ij} + c'_{ij}) \wedge 1]$
$(f_{ij}, f'_{ij}) = [(d_{ij} + c_{ij} - 1) \vee 0, (d'_{ij} + c'_{ij}) \wedge 1]$
$(g_{ij}, g'_{ij}) = [(a_{ij} + e_{ij} - 1) \vee 0, (a'_{ij} + e'_{ij}) \wedge 1]$
To prove $(f_{ij}, f'_{ij}) = (g_{ij}, g'_{ij})$.

Case 1 : If $a_{ij} + b_{ij} - 1 \leq 0$ and $a'_{ij} + b'_{ij} < 1$ then
$(f_{ij}, f'_{ij}) = [0, (a'_{ij} + b'_{ij} + c_{ij}) \wedge 1]$.

Sub case 1.1 : If $a'_{ij} + b'_{ij} + c'_{ij} < 1$ then $(f_{ij}, f'_{ij}) = (0, a'_{ij} + b'_{ij} + c'_{ij})$ either
$b'_{ij} + c'_{ij} < 1$ and $b_{ij} + c_{ij} - 1 < 0$
$\Rightarrow (g_{ij}, g'_{ij}) = [0, a'_{ij} + b'_{ij} + c'_{ij}] = (f_{ij}, f'_{ij})$.
If $b_{ij} + c_{ij} - 1 > 0$ and $b'_{ij} + c_{ij} < 1$ then
$(g_{ij}, g'_{ij}) = [(a_{ij} + b_{ij} + c_{ij} - 2) \vee 0, (a'_{ij} + b_{ij} + c'_{ij}) \wedge 1]$
$= [0, a'_{ij} + b'_{ij} + c'_{ij}] = (f_{ij}, f'_{ij})$.

Sub case 1.2 : If $a'_{ij} + b'_{ij} + c'_{ij} > 1$ then $(f_{ij}, f'_{ij}) = (0, 1)$.
If $b_{ij} + c_{ij} - 1 < 0$ and $b'_{ij} + c'_{ij} > 1$ then $(g_{ij}, g'_{ij}) = (0, 1) = (f_{ij}, f''_{ij})$.
If $b_{ij} + c_{ij} - 1 < 0$ and $b'_{ij} + c'_{ij} < 1$ then $(g_{ij}, g'_{ij}) = (0, 1) = (f_{ij}, f''_{ij})$.
If $b_{ij} + c_{ij} - 1 > 0$ and $b'_{ij} + c'_{ij} > 1$ then
$(g_{ij}, g'_{ij}) = [(a_{ij} + b_{ij} + c_{ij} - 2) \vee 0, 1] = (0, 1) = (f_{ij}, f'_{ij})$.
If $b_{ij} + c_{ij} - 1 > 0$ and $b'_{ij} + c'_{ij} < 1$.
Clearly from the above it is true.

Case 2 : If $a_{ij} + b_{ij} - 1 < 0$ and $a'_{ij} + b'_{ij} \geq 1$ then $(f_{ij}, f'_{ij}) = (0, 1)$.
Using 1.1 and 1.2 clearly it is true.

Case 3 : If $a_{ij} + b_{ij} - 1 > 0$ and $a'_{ij} + b'_{ij} \leq 1$ then
$(f_{ij}, f'_{ij}) = [(a_{ij} + b_{ij} + c_{ij} - 2) \vee 0, (a'_{ij} + b'_{ij} + c'_{ij}) \wedge 1]$.

Sub case 3.1 : If $a_{ij} + b_{ij} + c_{ij} - 2 < 0$ and $a'_{ij} + b'_{ij} + c'_{ij} < 1$ then
$(f_{ij}, f'_{ij}) = [0, a'_{ij} + b'_{ij} + c'_{ij}]$.
For any values of $(b_{ij}, b'_{ij}), (c_{ij}, c'_{ij}), (f_{ij}, f'_{ij}) = (g_{ij}, g'_{ij})$.

Sub case 3.2 : If $a_{ij} + b_{ij} + c_{ij} - 2 \geq 0$ and $a'_{ij} + b'_{ij} + c'_{ij} < 1$ then
$(f_{ij}, f'_{ij}) = [a_{ij} + b_{ij} + c_{ij} - 2, a'_{ij} + b'_{ij} + c'_{ij}]$.
Since $a_{ij} + b_{ij} + c_{ij} - 2 = (a_{ij} - 1) + (b_{ij} + c_{ij} - 1) > 0$
$\Rightarrow b_{ij} + c_{ij} - 1 \geq 0$
$(g_{ij}, g'_{ij}) = [a_{ij} + b_{ij} + c_{ij} - 2, a'_{ij} + b'_{ij} + c'_{ij}] = (f_{ij}, f'_{ij})$.

Sub case 3.3 : If $a_{ij} + b_{ij} + c_{ij} - 2 < 0$ and $a'_{ij} + b'_{ij} + c'_{ij} > 1$.
Clearly $f_{ij} = g_{ij}$ and $f'_{ij} = 1$.
If $b'_{ij} + c'_{ij} > 1 \Rightarrow g'_{ij} = 1$.
If $b'_{ij} + c'_{ij} \leq 1 \Rightarrow g'_{ij} = 1$.

Case 4 : If $a_{ij} + b_{ij} - 1 > 0$ and $a'_{ij} + b_{ij} \geq 1$ then
$(f_{ij}, f'_{ij}) = [(a_{ij} + b_{ij} + c_{ij} - 2) \vee 0, 1]$
from case 3 $f_{ij} = g_{ij}$.

Sub case 4.1 : If $b'_{ij} + c'_{ij} \geq 1$ then $g'_{ij} = 1 = f'_{ij}$ or if $b'_{ij} + c'_{ij} < 1$ then $g'_{ij} = 1 = f'_{ij}$.
From all the above cases $(f_{ij}, f'_{ij}) = (g_{ij}, g'_{ij})$. □

Proposition 3.7 : (i)$(\mathscr{F}_{mn}, \oplus, O)$ is a commutative monoid.
(ii)$(\mathscr{F}_{mn}, \odot, J)$ is also a commutative monoid.

Proof : It is clear from the Defnition 2.9 and Definition 2.10 and the Proposition 3.6. □

Corollary 3.8 : consider the two monoids $(\mathscr{F}_{mn}, \oplus, O)$ and $(\mathscr{F}_{mn}, \odot, J)$ and a function $f : \mathscr{F}_{mn} \to \mathscr{F}_{mn}$ such that $f(A) = A^c$,then there exists a monoid homomorphism under the two lukasiwicz conjunction and disjunction operators.

Proof : consider the two elements $A, B \in \mathscr{F}_{mn}$. Now

$$\begin{aligned} f[A \oplus B] &= f[(a_{ij} + b_{ij}) \wedge 1, (a'_{ij} + b'_{ij} - 1) \vee 0] \\ &= [(a'_{ij} + b'_{ij} - 1) \vee 0, (a_{ij} + b_{ij}) \wedge 1] = [(a'_{ij}, a_{ij}) \odot (b'_{ij}, b_{ij})] \\ &= A^c \odot B^c = f(A) \odot f(B) \end{aligned}$$

therefore $f(A \oplus B) = f(A) \odot f(B)$
and also $f(O) = O^c = J$,$f(J) = J^c = O$ where O and J are the identities.
From the Definition 2.11 there exists a monoid homomorphism. □

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