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CHARACTERIZATION OF PRIME FILTERS IN $(\mathcal{Z}^+, \leq_C)$

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Abstract

A convolution is a mapping $\mathcal{C}$ of the set Z^+ of positive integers into the set $\mathcal{P}(Z^+)$ of all subsets of Z^+ such that, for any $n \in Z^+$, each member of $C(n)$ is a divisor of n. If $D(n)$ is the set of all divisors of n, for any n, then D is called the Dirichlet's convolution [2]. If $U(n)$ is the set of all Unitary (square free) divisors of n , for any n, then U is called unitary (square free) convolution. Corresponding to any general convolution C, we can define a binary relation $\leq_C$ on Z^+ by '$m \leq_C n$ if and only if $m \in C(n)$ '. In this paper, we present a characterization for the prime filters in $(\mathcal{Z}^+, \leq_\mathcal{C})$, where $\leq_\mathcal{C}$ is the binary relation induced by the convolution $\mathcal{C}$.

1. Introduction

A convolution is a mapping $\mathcal{C}$ of the set $\mathcal{Z}^+$ of positive integers into the set $\mathcal{P}(\mathcal{Z}^+)$ of subsets of $\mathcal{Z}^+$ such that, for any $n \in \mathcal{Z}^+$, $\mathcal{C}n$ is a nonempty set of divisors of n. If $\mathcal{C}(n)$ is the set of all divisors of n, for each $n \in \mathcal{Z}^+$, then $\mathcal{C}$ is the classical Dirichlet convolution [2]. If $\mathcal{C}(n) = (\{d/d|n \text{ and } (d, \frac{n}{d}) = 1\})$, then $\mathcal{C}$ is the Unitary convolution

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[1]. As another example if Ç(n)=$\{d/d|n$ and m^k does not *divide* d for any $m \in \mathcal{Z}^+\}$ then $\mathcal{C}$ is the k-free convolution.

$$\mathcal{C}(n) = \{d/d|n \text{ and } (d, \frac{n}{d}) = 1\}.$$

Corresponding to any convolution $\mathcal{C}$, we can define a binary relation $\leq_{\mathcal{C}}$ in a natural way by

$$(m \leq_{\mathcal{C}} n) \text{ if and only if } m \in \mathcal{C}(n).$$

) $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ and is called partial order induced by the convolution $\mathcal{C}$ ([6], [7]). In this paper, we discuss filters in $(\mathcal{N}, \leq_C^p)$ and characterization of prime filters of $(\mathcal{Z}^+, \leq_C)$ in terms of those of $(\mathcal{N}, \leq_C^p)$.

2. Preliminaries

Let us recall that a partial order on a non-empty set X is defined as a binary relation $\leq$ on X which is reflexive $(a \leq a)$, transitive $(a \leq b, b \leq c \Longrightarrow a \leq c)$ and antisymmetric $(a \leq b, b \leq a \Longrightarrow a = b)$ and that a pair $(X, \leq)$ is called a partially ordered set(poset) if X is a non-empty set and $\leq$ is a partial order on X. For any $A \subseteq X$ and $x \in X$, x is called a lower(upper) bound of A if $x \leq a$(respectively $a \leq x$) for all $a \in A$. We have the usual notations of the greatest lower bound(glb) and least upper bound(lub) of A in X. If A is a finite subset $\{a_1, a_2, \cdots, a_n\}$, the glb of A(lub of A) is denoted by $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ or $\bigwedge_{i=1}^{n} a_i$ (respectively by $a_1 \vee a_2 \vee \cdots \vee a_n$ or $\bigvee_{i=1}^{n} a_i$). A partially ordered set $(X, \leq)$ is called a meet semi lattice if $a \wedge b$ (=glb$\{a, b\}$) exists for all a and $b \in X$. $(X, \leq)$ is called a join semi lattice if $a \vee b$ (=lub$\{a, b\}$) exists for all a and $b \in X$. A poset $(X, \leq)$ is called a lattice if it is both a meet and join semi lattice. Equivalently, lattice can also be defined as an algebraic system $(X, \wedge, \vee)$, where $\wedge$ and $\vee$ are binary operations which are associative, commutative and idempotent and satisfying the absorption laws, namely $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ for all $a, b \in X$; in this case the partial order $\leq$ on X is such that $a \wedge b$ and $a \vee b$ are respectively the glb and lub of $\{a, b\}$. The algebraic operations $\wedge$ and $\vee$ and the partial order $\leq$ are related by

$$a = a \wedge b \iff a \leq b \iff a \vee b = b.$$

Throughout the paper, $\mathcal{Z}^+$ and $\mathcal{N}$ denote the set of positive integers and the set of non-negative integers respectively.

Definition 1 : A mapping $\mathcal{C} : \mathcal{Z}^+ \longrightarrow \mathcal{P}(\mathcal{Z}^+)$ is called a convolution if the following are satisfied for any $n \in \mathcal{Z}^+$.

(1) $\mathcal{C}(n)$ is a set of positive divisors of n

(2) $n \in \mathcal{C}(n)$

(3) $\mathcal{C}(n) = \bigcup_{m \in \mathcal{C}(n)} \mathcal{C}(m)$.

Definition 2 : For any convolution $\mathcal{C}$ and m and $n \in \mathcal{Z}^+$, we define

$$m \leq n \text{ if and only if } m \in \mathcal{C}(n)$$

Then $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ and is called the partial order induced by $\mathcal{C}$ on $\mathcal{Z}^+$. In fact, for any mapping $\mathcal{C} : \mathcal{Z}^+ \longrightarrow \mathcal{P}(\mathcal{Z}^+)$ such that each member of $\mathcal{C}(n)$ is a divisor of n, $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ if and only if $\mathcal{C}$ is a convolution [7], as defined above.

Definition 3 : For any subset A of $\mathcal{Z}^+$ and for any prime number p, let

$$A^p = \{ \ \theta(n)(p) \mid n \in A\}$$

Then A^p is a subset of N for each $p \in P$.

We have the following two theorems on filters in $(\mathcal{Z}^+, \leq_C)$ and $(\mathcal{N}, \leq_C^p)$.

Theorem 1 : Let F be a filter of $(\mathcal{Z}^+, \leq_C)$. Then F^p is a filter of $(\mathcal{N}, \leq_C^p)$ for each $p \in P$ and $F = \{ \ n \in \mathcal{Z}^+ \ \mid \ \theta(n)(p) \in F^p \text{ for all } p \in P \ \}$ [3].

Theorem 2 : Let F be the set of all filters of $(\mathcal{Z}^+, \leq_C)$ and F^p be that of $(\mathcal{N}, \leq_C^p)$ for each $p \in P$. Let

$$\sum_{p \in P} \mathcal{F}^p = \{ \ f : P \longrightarrow \bigcup_{p \in P} \mathcal{F}^p \text{ and } f(p) = \mathcal{N} \text{ for all but finite number of } p'\text{s} \ \}$$

Then $\sum_{p \in P} \mathcal{F}^p$ is a partially ordered set with respect to the partial order defined by

$$f \leq g \text{ if and only if } f(p) \subseteq g(p) \text{ for all } p \in P$$

and $\mathcal{F}$ is order isomorphic with $\sum_{p \in P} \mathcal{F}^p$ [3].

3. Prime Filters in $(\mathcal{Z}^+, \leq_C)$

Definition 4 : Let $(S, \wedge)$ be a meet semi lattice. A proper filter F of S is called a **prime filter** if, for any a and b in S,

$$a \vee b \text{ exists in } S \text{ and } a \vee b \in F \implies a \in F \text{ or } b \in F.$$

Note that the concept of prime filter is not just the dual of a prime ideal in a meet semi lattice. Recall that a proper ideal I is prime if and only if, for any ideals J and K,

$$J \cap K \subseteq I \implies J \subseteq I \text{ or } K \subseteq I.$$

However, we have the following.

Theorem 3 : Let F be a proper filter of a meet semi lattice $(S, \wedge)$ satisfying the property that, for any filters G and H of S,

$$G \cap H \subseteq F \implies G \subseteq F \quad \text{or} \quad H \subseteq F.$$

Then F is a prime filter.

Proof : Let a and $b \in S$ such that $a \vee b$ exists and $a \vee b \in F$. Then, consider the principal filters $[a)$ and $[b)$. We have

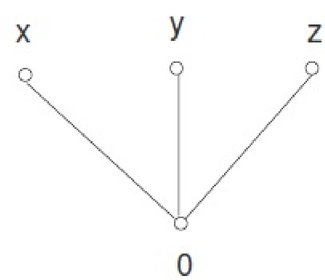
$$[a) \cap [b) = [a \vee b) \subseteq F.$$

and, from the hypothesis, $[a) \subseteq F$ or $[b) \subseteq F$ so that $a \in F$ or $b \in F$.

Thus F is a prime filter.

The converse of the above theorem is not true in general. For, consider the following.

Example 1 : Consider the semi lattice $(S, \wedge)$ whose Hasse diagram is given below.



Let $F = [x) = \{x\}$. If a and $b \in S$ such that $a \vee b$ exists and $a \vee b \in F$, then $a \vee b = x$ and hence one of a and b must be x (Note that $x \vee y, y \vee z, x \vee z$ do not exist in S). Therefore F is a prime filter. But,

$$[y) \cap [z) = \emptyset \subseteq F \text{ and } [y) \nsubseteq F \text{ and } [z) \nsubseteq F.$$

Even though the converse of theorem 3. is not true in a meet semi lattice, this is true in the case of a lattice.

Theorem 4 : Let $(L, \wedge, \vee)$ be a lattice and F a proper filter of L. Then F is a prime filter if and only if, for any filters G and H in L,

$$G \cap H \subseteq F \implies G \subseteq F \text{ or } H \subseteq F.$$

Proof : Suppose that F is a prime filter and G and H are filters of L such that $G \nsubseteq F$ and $H \nsubseteq F$. Then, we can choose elements $a \in G$ and $a \in H$ such that $a \notin F$ and $b \notin F$. Since F is prime, we have $a \vee b \notin F$.

But $a \vee b \in G$ and $a \vee b \in H$ and hence $a \vee b \in G \cap H$. Therefore $G \cap H \nsubseteq F$. The converse is proved in Theorem 3.

From the above theorem, it follows that a proper filter F of a lattice L is prime if and only if F is a prime element in the lattice of filters of L.

Definition 5 : Let $(S, \wedge)$ be a meet semi lattice with smallest element 0 and let $0 \neq x \in S$. x is said to be join-irreducible and y and $z \in S$ and $x = y \vee z \Longrightarrow x = y \; or \; x = z$.

Theorem 5 : Let x be any element in a meet semi lattice $(S, \wedge)$. If $[x)$ is a prime filter of S, then x is join-irreducible.

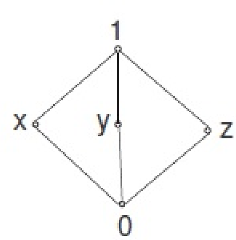
Proof : Suppose that x is not join-irreducible. Then there exist elements y and z such that

$$y < x, \; z < x \text{ and } y \vee z \text{ exists and equals to } x.$$

Now, $y \vee z \in [x)$ and $y \notin [x)$ and $z \notin [x)$ and hence $[x)$ is not a prime filter.

The converse of the theorem is not true, even in the case of lattices. For, consider the example given below.

Example 2 : Let $(L, \wedge, \vee)$ be the lattice whose Hasse diagram is given below.



Here x is join-irreducible(since 0 is the only element which is strictly less than x). But $y \vee z = 1 \in [x)$ and $y \notin [x)$ and $z \notin [x)$ and hence $[x)$ is not a prime filter.

However, in the case of distributive lattices, we have the following theorem.

Theorem 6 : Let $(L, \wedge, \vee)$ be a distributive lattice and $x \in L$. Then $[x)$ is a prime filter if and only if x is join-irreducible.

Proof : Note that $[x) = L$ if and only if x is the smallest element of L. Suppose that x is join-irreducible. Then $x \neq 0$ and hence $[x)$ is a proper filter. If $y \vee z \in [x)$, then $x \leq y \vee z$ and therefore

$$x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Since x is join-irreducible, $x = x \wedge y$ or $x = x \wedge z$ and therefore $y \in [x)$ or $z \in [x)$. Thus $[x)$ is a prime filter. The converse is proved in Theorem 5.

Now, we shall determine all the prime filters of the meet semi lattice $(\mathcal{Z}^+, \leq_C)$ where C is a multiplicative convolution which is closed under finite intersections.

We have the following theorem on irreducible elements in meet semi lattice $(\mathcal{Z}^+, \leq_C)$.

Theorem 7 : Let C be a multiplicative convolution such that $(\mathcal{Z}^+, \leq_C)$ is meet semi lattice and $x \in \mathcal{Z}^+$. Then x is join-irreducible in $(\mathcal{Z}^+, \leq_C)$ if and only if $x = p^a$ for some prime number p and a join-irreducible element a in $(\mathcal{N}, \leq_C^p)$ [4] [5].

Theorem 8 : Let F be a prime filter of $(\mathcal{Z}^+, \leq_C)$. Then $F = [p^a)$ for some prime number p and a positive integer a which is join-irreducible in $(\mathcal{N}, \leq_C^p)$.

Proof : By hypothesis, F is a prime filter. That is, there exists $x \in \mathcal{Z}^+$ such that $F = [x)$. By Theorem 5, x is join-irreducible. Also, by Theorem 7 , $x = p^a$ for some prime number p and a join-irreducible element a in $(\mathcal{N}, \leq_C^p)$. Thus $F = [p^a)$.

The converse of the above theorem is not true, even when $(\mathcal{Z}^+, \leq_C)$ is a lattice. For, consider the following.

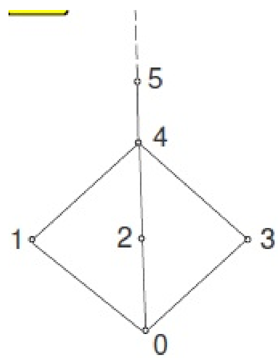
Example 3 : For any prime number p and $a \in \mathcal{N}$, define

$$C(p^a) = \begin{cases} \{1, p^a\} & \text{if } a < 4 \\ \{1, p, p^2, \cdots, p^a\} & \text{if } a \geq 4 \end{cases}$$

and extend C to $\mathcal{Z}^+$ multiplicatively; that is

$$C(\prod_{i=1}^{r} p_i^{a_i}) = \prod_{i=1}^{r} C(p_i^{a_i})$$

for any distinct primes $p_1, p_2, \cdots, p_r$ and $a_1, a_2, \cdots, a_r \in \mathcal{N}$
Then C is a multiplicative convolution such that $(\mathcal{Z}^+, \leq_C)$ is a lattice.
The Hasse diagram for $(\mathcal{N}, \leq_C^p)$ is given below, for any prime number p.



Clearly 1 is join-irreducible in $(\mathcal{N}, \leq_C^2)$. But $[2^1)$ is not a prime filter, since $2^2 \vee 2^3 = 2^4 \in [2^1)$, $2^2 \notin [2^1)$ and $2^3 \notin [2^1)$
However, we have the following

Theorem 9 : Suppose that $(\mathcal{Z}^+, \leq_C)$ is a distributive lattice and F a filter of $(\mathcal{Z}^+, \leq_C)$. Then F is a prime filter if and only if $F = [p^a)$ where p is a prime number and a is join irreducible in $(\mathcal{N}, \leq_C^p)$.

Proof : This follows from Theorem 6 and Theorem 7.
In the following we get another characterization of prime filters of $(\mathcal{Z}^+, \leq_C)$ in terms of those of $(\mathcal{N}, \leq_C^p)$.
For any filter F of $(\mathcal{Z}^+, \leq_C)$ and for any $p \in P$, we define

$$F^p = \{\ \theta(n)(p) \ \mid\ n \in F\ \}$$

where $\theta(n)(p)$ is the largest $a \in \mathcal{N}$ such that p^a divides n

Theorem 10 : A filter $(\mathcal{Z}^+, \leq_C)$ is prime if and only if there exists unique $p \in P$ such that F^p is a prime filter of $(\mathcal{N}, \leq_C^p)$ and $F^q = \mathcal{N}$ for all $q \neq p$ in P and, in this case,

$$F = \{\ n \in \mathcal{Z}^+ \ \mid\ \theta(n)(p) \in F^p\ \}.$$

Proof : Suppose that F is a prime filter of $(\mathcal{Z}^+, \leq_C)$. Then, by Theorem 8, $F = [p^a)$ for some $p \in P$ and for some $a \in \mathcal{N}$. For this p, we prove that F^p is a prime filter of $(\mathcal{N}, \leq_C^p)$. Also,

$$\begin{aligned} F^p &= \{\ \theta(n)(p)\ \mid\ n \in F\ \} \\ &= \{\ \theta(n)(p)\ \mid\ p^a \leq_C n\ \} \\ &= \{\ b \in \mathcal{N}\ \mid\ a \leq_C^p b\ \} = [a) \text{ in } (\mathcal{N}, \leq_C^p) \end{aligned}$$

Since $a > 0$, $[a)$ and hence F^p is a proper filter of $(\mathcal{N}, \leq_C^p)$.
Observe that, for any $m \in \mathcal{Z}^+$ such that p does not divide m, we have $p^a \wedge m = 1$ and, since $p^a \in F$ and F is a proper filter of $(\mathcal{Z}^+, \leq_C)$, we get that $m \notin F$. Let b and $c \in \mathcal{N}$ such that $b \vee c$ exists in $(\mathcal{N}, \leq_C^p)$ and $b \vee c \in F^p = [a)$. Then $b \vee c = \theta(n)(p)$ for some $n \in F$. Let us write $n = p^{b\vee c}.m$, where $m \in \mathcal{Z}^+$ such that $(p, m) = 1$. Since $b \vee c$ exists in $(\mathcal{N}, \leq_C^p)$, it follows that $p^b \vee p^c$ exists in $(\mathcal{Z}^+, \leq_C)$ and is equal to $p^{b\vee c}$. Also, since $(p, m) = 1$, $(p^{b\vee c}, m)$ is also 1 and hence $p^{b\vee c}.m = n$. Therefore

$$p^b \vee p^c \vee m = n \in F$$

Since $(p, m) = 1$, p does not divide m and hence $m \notin F$. Since F is prime, $p^b \in F$ or $p^c \in F$ and therefore $b \in F^p$ or $c \in F^p$. Thus F^p is prime.
Also, for any $p \neq q \in P$,

$$\begin{aligned} b \in \mathcal{N} &\implies p^a \leq_C p^a.q^b \\ &\implies p^a.q^b \in [p^a) = F \\ &\implies b = \theta(p^a.q^b)(q) \in F^q \end{aligned}$$

and hence $F^q = N$ for all $p \neq q \in P$. The uniqueness of p is trivial. Further, by Theorem 1,

$$F = \{\ n \in \mathcal{Z}^+\ \mid\ \theta(n)(q) \in F^q \text{ for all } q \in P\} = \{\ n \in \mathcal{Z}^+\ \mid\ \theta(n)(p) \in F^p\}$$

since $F^q = N$ for all $q \neq p$.
Conversely suppose that there exists $p \in P$ such that F^p is a prime filter of $(\mathcal{N}, \leq_C^p)$ and $F^q = N$ for all $p \neq q \in N$. Let m and $n \in \mathcal{Z}^+$ such that $m \vee n$ exists in $(\mathcal{Z}^+, \leq_C)$ and $m \vee n \in F$. Then $\theta(m)(p) \vee \theta(n)(p)$ exists in $(\mathcal{N}, \leq_C^p)$ and is equal to $\theta(m \vee n)(p) \in F^p$. Since F^p is a prime filter, either $\theta(m)(p) \in F$ or $\theta(n)(p) \in F$. Since $F^q = N$ for all $q \neq p \in N$, we get that

$$\begin{aligned} &\theta(m)(q) \in F^q \text{ for all } q \in P \\ \text{or } &\theta(n)(q) \in F^q \text{ for all } q \in P. \end{aligned}$$

Since $F = \{k \in \mathcal{Z}^+ \mid \theta(k)(q) \in F^q$ for all $q \in P$ $\}$, by Theorem 1, we have $m \in F$ or $n \in F$. Thus F is a prime filter of $(\mathcal{Z}^+, \leq_C)$.

Theorem 11 : Let $(S, \wedge)$ be any meet semi lattice. Then every proper filter of $(S, \wedge)$ is prime if and only if, for any x and y in S,

$$x \vee y \text{ exists in } S \Leftrightarrow\ x \text{ and } y \text{ are comparable.}$$

Proof : Suppose that every proper filter of $(S, \wedge)$ is prime. Let x and $y \in S$. If x and y are comparable, then clearly $x \vee y$ exists in S.On the other hand, suppose $x \vee y$ exists and $x \vee y = z$. If $[z) = S$, then x and $y \in [z)$ and hence $x = z = y$. If $[z) \neq S$, then by hypothesis, $[z)$ is a prime filter and $x \vee y \in [z)$ and hence $x \in [z)$ or $y \in [z)$ so that $x = z$ or $y = z$. Therefore $x = x \vee y$ or $y = x \vee y$, which imply that x and y are comparable. The converse is trivial.

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