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A COMPARATIVE STUDY OF FINITE VOLUME METHOD AND FINITE DIFFERENCE METHOD FOR ONE DIMENSIONAL DIFFUSION PROBLEM

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Abstract

Finite Volume and Finite difference solutions are widely employed in CFD. Both the methods are being in use for solving diffusion problems appearing in different branches of fluid engineering. The present paper deals with the description and comparison of the finite volume method and finite difference method for solving steady state diffusion equation in one dimensional domain.

1. Introduction

1.1 Finite Volume Method

The finite volume method is a method for representing and evaluating partial differential equations in the form of algebraic equations. Similar to the finite difference method or finite element method, values are calculated at discrete places on a meshed geometry. "Finite volume" refers to the small volume surrounding each node point on a mesh. In

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this method, volume integrals in a partial differential equation that contain a divergence term are converted to surface integrals, using the divergence theorem. These terms are then evaluated as fluxes at the surfaces of each finite volume. Because the flux entering a given volume is identical to that leaving the adjacent volume, this method is conservative. Hence, conservation of mass, momentum, energy is ensured at each cell/finite volume level. Another advantage of the finite volume method is that it is easily formulated to allow for unstructured meshes. The method is used in many computational fluid dynamics packages.

1.2 Finite Difference Method

In mathematics and engineering, finite-difference methods are numerical methods for approximating the solutions to differential equations using finite difference equations to approximate derivatives. The first step in developing a numerical solution for this method involves dividing the geometric domain into discrete nodal points. Then, all the derivatives in the given differential equations are written in terms of finite differences. The difference equations obtained is not same as the partial differential equation. The difference equation is an algebraic equation, which when written at all grid points in the domain yields a simultaneous system of algebraic equations. In turn, by some fashion, these algebraic equations are solved numerically for the dependent variable at all grid points. Hence, the general concept of a finite difference solution is to represent the governing partial differential equations by means of difference equations, and to solve these difference equations for numerical values of the dependent variables at each of the discrete grid points which cover the physical domain of interest. Finite difference (FD) methods are intuitive and easy to implement for simple problems. However, for complex problems like moving boundaries or an unstructured grid they are not easy to work with.

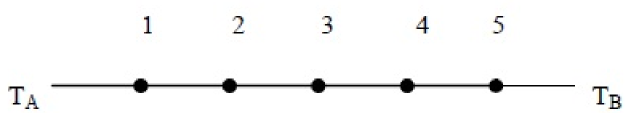
2. Solution of Real Life Problem

In this section, we use finite volume method and finite difference method for solving diffusion problem in one dimensional domain.

Consider the problem of heat conduction in an insulated rod of length 0.5 m, whose ends are maintained at constant temperatures 100^oC and 200^oC respectively. The one dimensional problem is governed by equation

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+q=0. \tag{1}$$

Thermal conductivity is k and uniform heat generation is q. The cross sectional area A is 0.01 m^2. Let us divide the length of the rod into five equal control volumes as shown in figure



We discuss two cases here for $q=0$ and $q=1000$.

Case 1 : When $q=0kW/m^3$ (In the absence of any source)

Finite volume method : The governing equation is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \tag{2}$$

where $k=1000W/m.K$. In this case, for each of the nodes 2, 3 and 4 temperature values to the east and west are available as nodal values. The integration of the governing equation (2) over the control volume, gives the discretised equation for the nodal points 2, 3 and 4 as

$$a_pT_p=a_wT_w+a_ET_E$$

where

a_w	a_E	a_p
$\frac{kA}{\delta x}$	$\frac{kA}{\delta x}$	a_w+a_E

Since nodes 1 and 5 are boundary nodes, they require special attention. Integration of the governing equation at node 1 gives discretised equation as

$$a_pT_p=a_wT_w+a_ET_e+S_u$$

with

a_w	a_E	a_p	S_p	S_u
0	$\frac{kA}{\delta x}$	$a_w + a_E - S_p$	$-\frac{2kA}{\delta x}$	$\frac{2kA}{\delta x}T_A$

In the same way the discretised equation at node 5 is given as

$$a_pT_p = a_wT_w + a_ET_E + S_u$$

with

a_w	a_E	a_p	S_p	S_u
$\frac{kA}{\delta x}$	0	$a_w + a_E - S_p$	$-\frac{2kA}{\delta x}$	$\frac{2kA}{\delta x}T_B$

Taking $k = 1000W/m.K$, the numerical value of the coefficient of each discretised equation can easily be worked out. The resulting set of algebraic equations in his case is

$$\begin{aligned} 300T_1 &= 100T_2 + 200T_A \\ 200T_2 &= 100T_1 + 100T_3 \\ 200T_3 &= 100T_2 + 100T_4 \\ 200T_4 &= 100T_3 + 100T_5 \\ 300_5 &= 100T_4 + 200T_B. \end{aligned}$$

This set of equations can be rearranged as

$$\left\{\begin{array}{ccccc} 300 & -100 & 0 & 0 & 0 \\ -100 & 200 & -100 & 0 & 0 \\ 0 & -100 & 200 & -100 & 0 \\ 0 & 0 & -100 & 200 & -100 \\ 0 & 0 & 0 & -100 & 300 \end{array}\right\} \left\{\begin{array}{c} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{array}\right\} = \left\{\begin{array}{c} 200T_A \\ 0 \\ 0 \\ 0 \\ 200T_B \end{array}\right\}.$$

Solving , we get

$$\left\{\begin{array}{c} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{array}\right\} = \left\{\begin{array}{c} 110 \\ 130 \\ 150 \\ 170 \\ 190 \end{array}\right\}$$

Finite Difference Method : The discretized form of Eqn. (2) can be written as

$$T_{i-1} - 2T_i + T_{i+1} = 0.$$

The resulting set of algebraic equations for this example is

$$\begin{aligned} T_A - 2T_1 + T_1 &= 0 \\ T_1 - 2T_2 + T_3 &= 0 \\ T_2 - 2T_3 + T_4 &= 0 \\ T_3 - 2T_4 + T_5 &= 0 \\ T_4 - 2T_5 + T_B &= 0 \end{aligned}$$

This set of equations can be rearranged as

$$\left\{\begin{array}{ccccc} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right\} \left\{\begin{array}{c} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{array}\right\} = \left\{\begin{array}{c} -T_A \\ 0 \\ 0 \\ 0 \\ -T_B \end{array}\right\}.$$

Solving which, we get

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 116.66 \\ 133.33 \\ 150 \\ 166.66 \\ 183.33 \end{Bmatrix}$$

Analytical Solution : It is obtained using linear distribution between the specified boundary temperature: $T = 200x + 100$. Comparison between FVM and FDM with the Analytical solution is given in Table 1.

Table 1 : Comparative study when $q = 0$

Node	Distance	Finite volume solution (FVM)	Finite difference solution (FDM)	Analytical solution	% age Error in FVM	% age error in FDM
1	0.05	110	116.66	110	0	6.05
2	0.15	130	133.33	130	0	2.56
3	0.25	150	150	150	0	0
4	0.35	170	166.66	170	0	-1.96
5	0.45	190	183.33	190	0	-3.51

Case 2 : When $q = 1000kW/m3$. The governing equation is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + 1000 = 0. \tag{3}$$

We will discuss the case for two values of thermal conductivity, for $k_1 = 0.5W/m.\ K$ and $k_2 = 1000W/m.K$.

Finite Volume Method : The formal integration of the governing equation over the control volume gives the discretised equation for the nodal points 2,3 and 4 as

$$a_pT_p = a_wT_w + a_ET_E + S_u$$

with

a_w	a_E	a_p	S_p	S_u
$\frac{k}{\delta x}A$	$\frac{k}{\delta x}A$	$a_w + a_E - S_p$	0	$qA\delta x$

Integration of the governing equation at node 1 gives discretised equation as

$$a_p T_p = a_w T_w + a_E T_E + S_u$$

with

a_w	a_E	a_p	S_p	S_u
0	$\frac{k}{\delta x}A$	$a_w + a_E - S_p$	$-\frac{2kA}{\delta x}T_A$	$qA\delta x + \frac{2kA}{\delta x}T_A$

In the same way the discretised equation at node 5 is given as

$$a_p T_p = a_w T_w + a_E T_E + S_u$$

with

a_w	a_E	a_p	S_p	S_u
$\frac{k}{\delta x}A$	0	$a_w + a_E - S_p$	$-\frac{2kA}{\delta x}$	$qA\delta x + \frac{2kA}{\delta x}T_B$

The resulting set of algebraic equations for $k_1 = 0.5$ is

$$\begin{aligned}
0.15T_1 &= 0.05T_2 + 0.1T_A + 1 \\
0.10T_2 &= 0.05T_1 + 0.05T_3 + 1 \\
0.10T_3 &= 0.05T_2 + 0.0T_4 + 1 \\
0.10T_4 &= 0.05T_3 + 0.05T_5 + 1 \\
0.15T_5 &= 0.05T_4 + 0.01T_B + 1.
\end{aligned}$$

This set of equations can be rearranged as

$$\begin{Bmatrix} 0.15 & -0.05 & 0 & 0 & 0 \\ -0.05 & 0.10 & -0.05 & 0 & 0 \\ 0 & -0.05 & 0.1 & -0.05 & 0 \\ 0 & 0 & -0.05 & 0.1 & -0.05 \\ 0 & 0 & 0 & -0.05 & 0.15 \end{Bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 0.1T_A + 1 \\ 1 \\ 1 \\ 1 \\ 0.1T_B + 1 \end{Bmatrix}.$$

which after solving givest

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 135 \\ 185 \\ 215 \\ 225 \\ 215 \end{Bmatrix}$$

Finite Difference Method : The discretised form of Eqn. (3) can be written as

$$T_{i-1} - 2T_i + T_{i+1} = -20$$

The resulting set of algebraic equations for $k_1 = 0.5$ is

$$\begin{aligned} T_A - 2T_1 + T_2 &= -20 \\ T_1 - 2T_2 + T_3 &= -20 \\ T_2 - 2T_3 + T_4 &= -20 \\ T_3 - 2T_4 + T_5 &= -20 \\ T_4 - 2T_5 + T_B &= -20. \end{aligned}$$

This set of equations can be rearranged as

$$\begin{Bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{Bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} -120 \\ -20 \\ -20 \\ -20 \\ -220 \end{Bmatrix}.$$

Solving we get

$$\left\{\begin{array}{c} T_1 \\ \\ T_2 \\ \\ T_3 \\ \\ T_4 \\ \\ T_5 \end{array}\right\} = \left\{\begin{array}{c} 166.66 \\ \\ 213.33 \\ \\ 240.0 \\ \\ 246.66 \\ \\ 233.33 \end{array}\right\}$$

Analytical Solution : Analytical Solution for this model may be obtained by integrating the diffusion equation (3) twice with respect to x and y by subsequent application of the boundary conditions. This gives

$$T = \left[\frac{T_B - T_A}{L} + \frac{1000}{2k}(-x)\right] x + T_A.$$

The comparison between the finite volume solution ,finite difference method and the exact Analytical solution is shown in Table 2.

Table 2 : Comparative study when $q = 1000, k_l = 0.5$

Node	Distance	Finite volume solution (FVM)	Finite difference solution (FDM)	Analytical solution	% age Error in FVM	% age error in FDM
1	0.05	135	116.66	132.5	1.88	25.78
2	0.15	185	213.33	182.5	1.36	16.89
3	0.25	215	240	212.5	1.17	12.94
4	0.35	225	246.66	222.5	1.12	10.85
5	0.45	215	233.33	212.5	1.17	9.8

Now taking $k_2 = 1000$ the algebraic equations using Finite volume method are

$$\begin{aligned} 100T_1 &= 100T_2 + 200T_A + 1 \\ 200T_2 &= 1005T_1 + 100T_3 + 1 \\ 200T_3 &= 100T_2 + 100T_4 + 1 \\ 200T_4 &= 100T_3 + 100T_5 + 1 \\ 300T_5 &= 100T_4 + 200T_B + 1. \end{aligned}$$

Solving this set of equations we get

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 87.49 \\ 112.51 \\ 137.52 \\ 162.52 \\ 187.51 \end{Bmatrix}$$

Using Finite difference method, the set of algebraic equations is

$$\begin{aligned} T_A - 2T_1 + T_2 &= -0.01 \\ T_1 - 2T_2 + T_3 &= -0.01 \\ T_2 - 2T_3 + T_4 &= -0.01 \\ T_3 - 2T_4 + T_5 &= -001 \\ T_4 - 2T_5 + T_B &= -0.01. \end{aligned}$$

which after solving gives

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 116.68 \\ 133.36 \\ 150.025 \\ 166.68 \\ 183.34 \end{Bmatrix}$$

Using

$$T = \left[\frac{T_B - T_A}{L} + \frac{q}{2k}(L - x)\right] x + T_A$$

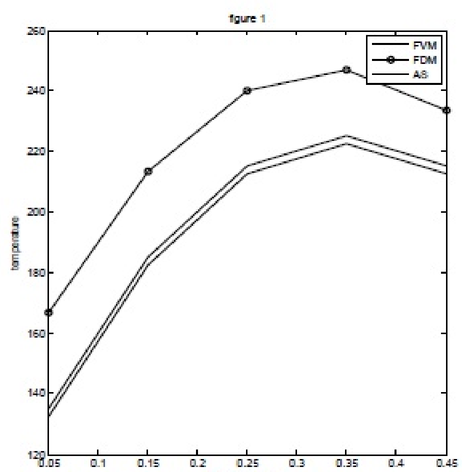
the Analytical solution for this model is obtained. The comparison between the finite volume solution, finite difference method and the exact Analytical solution is shown in table given below:

Table 3 : Comparative study when $q = 1000, k_2 = 1000$

Node	Distance	Finite volume solution (FVM)	Finite difference solution (FDM)	Analytical solution	% age Error in FVM	% age error in FDM
1	0.05	87.49	116.68	110.01	20.47	5.716
2	0.15	112.51	133.36	130.025	13.47	2.56
3	0.25	137.52	150.025	150.031	8.33	-0.0039
4	0.35	162.52	166.68	170.075	4.44	-1.996
5	0.45	187.51	183.34	190.025	1.323	-3.517

3. Comparison and Conclusion

It is clear from Table 1 that in absence of any source ($q = 0$), Finite Volume method gives very good results with zero or negligible error, even in a coarse grid of five node points. We observe that Finite difference method also gives results with acceptable error.



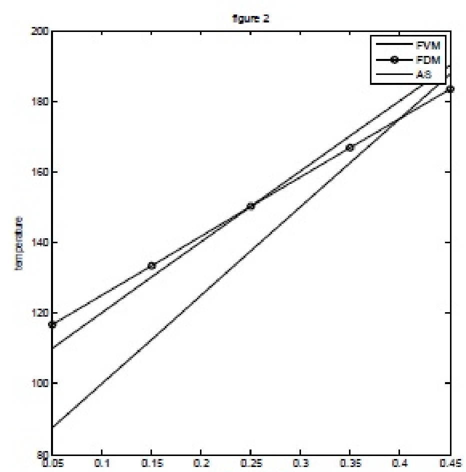


Figure 1 and 2 represents data from table 2 and 3 respectively both of which corresponds to case 2 when there is a source term in the governing equation. It is clear from fig. 1 that in the presence of heat source but low thermal conductivity finite volume method gives better results than finite difference method. But in case of high thermal conductivity (as shown in fig. 2) finite difference method is a better choice than finite volume method.

As far as numerical methods for solving fractional differential equations are concerned, finite difference methods were amongst the first developed. Later, Finite Volume Methods, which deal directly with equations in conservative form, were proposed. A key difference between the two methods is that the Finite Volume Method deals directly with the differential equation in conservative form, eliminating the need for product rule expansions in variable coefficient problems. Also, Finite Volume Method is more flexible than standard Finite Difference Method which mainly defined on the structured grid of simple domain.

We derived the finite difference and finite volume discretizations for steady state one dimensional diffusion equation and compared the numerical solution obtained with the two methods for several variable coefficient test problems. We conclude that the finite volume method is preferable for solving steady state one dimensional diffusion equation.

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