International J. of Pure & Engg. Mathematics (IJPEM)
ISSN 2348-3881, Vol. 4 No. I (April, 2016), pp. 107-116

GEOMETRIC MEAN DERIVATIVE-BASED CLOSED NEWTON-COTES QUADRATURE

T. RAMACHANDRAN[1], D. UDAYAKUMAR[2] AND R. PARMIALA[3]
[1] Department of Mathematics,
M.V. Muthiah Government Arts College for Women,
Dindigul - 624 001, Tamil Nadu, India
E-mail: yasrams@gmail.com
[2,3] Department of Mathematics,
Government Arts College (Autonomous),
Karur - 639 005, Tamil Nadu, India
E-mail: [2] drduk80@rediffmail.com, [3] rrparimala@gmail.com

Abstract

In this paper, a new set of Geometric mean derivative based closed Newton-cotes quadrature rule is derived for the evaluation of numerical integration, using the geometric mean value for the computation of derivative. It is also verified that this quadrature rule gives increase of single order of precision over the existing closed Newton-cotes formula and provide more accurate solution, when compared with the existing formula. Comparisons are made between the error terms of the proposed method and existing method and are discussed in detail. Finally, the superiority of the proposed rule is illustrated with a numerical examples.

1. Introduction

In Numerical Analysis, numerical integration constitutes a broad family of

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Key Words : *Numerical Integration, Closed Newton-Cotes formula, Geometric mean derivative, Numerical examples.*

2010 AMS Subject Classification : 65D30, 65D32.

algorithms for calculating the numerical value of a definite integral [1].It has several applications in the field of physics and Engineering. In the field of Mathematics, to get the high precision numerical integration formulas becomes one of the challenges [2].One of the popular method for the evaluation of numerical integration is quadrature rule is given by

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) \tag{1}$$

where there are $(n+1)$ distinct points $x_0 < x_1 < \cdots < x_n$ within the interval $[a,b]$, $x_i = x_0 + ih$, $i = 0,1,2,\cdots,n$, and $(n+1)$ weights $w_0, w_1, \cdots, w_n$. Many methods are available for deriving this w_i [1, 3, 12]. Select the values for $w_i, i = 0,1,\cdots,n, h = \frac{b-a}{n}$ so that the error of approximation for the method based on the precision of a quadrature formula is zero, that is

$$E_n[f] = \int_a^b f(x)dx - \sum_{i=0}^{n} w_i f(x_i) = 0 \quad \text{for} \quad f(x) = x^j,\ j = 0,1,\cdots,n. \tag{2}$$

Definition 1 : An integration method of the form (1) is said to be of order P, if it produces exact results ($E_n[f] = 0$) for all polynomials of degree less than or equal to P [13].

Several Closed Newton Cotes Quadrature formulas are derived so far by giving different values for n in the general quadrature formula. Some of the rules are as follows,

When $n = 1$: Trapezoidal rule

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a) + f(b)) - \frac{(b-a)^3}{12} f''(\xi), \quad \text{where} \quad \xi \in (a,b) \tag{3}$$

When $n = 2$: Simpson's 1/3rd rule

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi) \text{ where } \xi \in (a,b). \tag{4}$$

When $n = 3$: Simpson's 3/8th rule

$$\begin{aligned} \int_a^b f(x)dx &= \tfrac{b-a}{8}\left[f(a) + 3f\left(\tfrac{2a+b}{3}\right) + 3f\left(\tfrac{a+2b}{3}\right) + f(b)\right] \\ &\quad - \tfrac{(b-a)^5}{6480} f^{(4)}(\xi) \text{ where } \xi \in (a,b). \end{aligned} \tag{5}$$

When $n = 4$: Boole's rule

$$\begin{aligned} \int_a^b f(x)dx &= \tfrac{b-a}{90}\left[7f(a) + 32f\left(\tfrac{3a+b}{4}\right) + 12f\left(\tfrac{a+b}{2}\right) + 32f\left(\tfrac{a+3b}{4}\right) + 7f(b)\right] \\ &\quad - \tfrac{(b-a)^7}{1935360} f^{(6)}(\xi) \text{ where } \xi \in (a,b). \end{aligned} \tag{6}$$

It is known that the degree of precision is $n+1$ for even value of n and n for odd value of n.

The precision of a proposed formula have greater degree and are more accurate than the existing formula. There are so many works has been done on the improvement of Newton cotes formula. Dehghan et al., derived an improved formulas from the existing closed, semi-open, open, first and second kind of [6, 7, 8, 10, 11] Chebyshev Newton-cotes quadrature rules. Clarence O. E Burg and his companions introduced a new set of derivative based rules [4, 5, 9]. Weijing Zhao and Hongxing Li [15] introduced a new set of Midpoint derivative-based closed Newton-cotes quadrature rules.

Recently, we proposed [14] a new family of Midpoint derivative - based open Newton-cotes quadrature rules.

In this paper, Geometric Mean derivative-based closed Newton cotes quadrature formulas are presented, in which derivative value is included in addition to the existing function values and the remaining terms are formed as error terms. Numerical examples are discussed and the result shows that, the degree of precision of the new approach is higher than the existing Newton-cotes rules.

2. Geometric Mean Derivative-based Closed Newton Cotes

Quadrature rule : Geometric mean derivative based scheme for the Closed Newton Cotes formulas is derived here under for the evaluation of definite integral. In this method, the Geometric mean derivative is zero if either $a = 0$ or $b = 0$.

Theorem 1 : Closed Trapezoidal Rule ($n = 1$) using Geometric Mean derivative is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b)) - \frac{(b-a)^3}{12} f''(\sqrt{ab}). \tag{7}$$

The precision of this method is 2.

Proof : Since the rule (3) has the degree of precision 1. Now use the rule (7) for $f(x) = x^2$. When

$$f(x) = x^2, \quad \int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3);$$

$$[n-1] \Rightarrow \frac{b-a}{2}(a^2+b^2) - \frac{2(b-a)^3}{12} = \frac{1}{3}(b^3 - a^3).$$

It shows that the solution is Exact.

Therefore, the precision of Closed Trapezoidal Rule with Geometric Mean derivative is 2.

Theorem 2 : Closed Simpson's 1/3rd Rule with Geometric Mean derivative ($n = 2$) is

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\sqrt{ab}). \tag{8}$$

The precision of this method is 4.

Proof : Since the rule (4) has the degree of precision 3. Now use the rule (8) for $f(x) = x^4$. When

$$f(x) = x^4, \quad \int_a^b x^4 dx = \frac{1}{5}(b^5 - a^5);$$

$$[n-2] \Rightarrow \left(\frac{b-a}{6}\right)\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right] - \frac{24(b-a)^5}{2880} = \frac{1}{5}(b^5 - a^5).$$

It shows that the solution is Exact. Therefore, the precision of Closed Simpson's 1/3rd Rule with Geometric Mean derivative is 4.

Theorem 3 : Closed Simpson's 3/8rd Rule with Geometric Mean derivative ($n = 3$) is

$$\begin{aligned} &\textstyle\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3\left(\frac{2a+b}{3}\right)^4 + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] \\ &\quad - \textstyle\frac{(b-a)^5}{6480} f^{(4)}(\sqrt{ab}). \end{aligned} \tag{9}$$

The precision of this method is 4.

Proof : Since the rule (5) has the degree of precision 3. Now use the rule (9) for $f(x) = x^4$. When

$$f(x) = x^4, \quad \int_a^b x^4 dx = \frac{1}{5}(b^5 - a^5);$$

$$[n-3] \Rightarrow \left(\frac{b-a}{8}\right)\left[a^4 + 3\left(\frac{2a+b}{3}\right) + 3\left(\frac{a+2b}{3}\right)^4 + b^4\right] - \frac{24(b-a)^5}{6480} = \frac{1}{5}(b^5 - a^5).$$

It shows that the solution is Exact. Therefore, the precision of Closed Simpson's 3/8rd Rule with Geometric Mean derivative is 4.

Theorem 4 . Closed Boole's Rule with Geometric Mean derivative ($n = 4$) is

$$\begin{aligned} &\textstyle\int_a^b f(x)dx \approx \frac{b-a}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] \\ &\textstyle -\frac{(b-a)^7}{1935360} f^{(6)}(\sqrt{ab}). \end{aligned} \tag{10}$$

The precision of this method is 6.

Proof : Since the rule (6) has the degree of precision 5. Now use the rule (10) for $f(x) = x^6$. When

$$f(x) = x^6, \quad \int_a^b x^6 dx = \frac{1}{7}(b^7 - a^7);$$

$$[n-4] \Rightarrow \left(\tfrac{b-a}{90}\right)\left[7a^6 + 32\left(\tfrac{3a+b}{4}\right)^6 + 12\left(\tfrac{a+b}{2}\right)^6 + 32\left(\tfrac{a+3b}{4}\right)^6 + 7b^6\right]$$

$$+\tfrac{720(b-a)^7}{1935360} = \tfrac{1}{7}(b^7 - a^7).$$

It shows that the solution is Exact. Therefore, the precision of Closed Boole's Rule with Geometric Mean derivative is 6.

3. The Error Terms of Geometric Mean Derivative-based Closed Newton Cotes Quadrature Rule

The calculation of error terms is obtained by using the difference between the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$ and the exact result $\frac{1}{(p+1)!}\int_a^b x^{p+1}dx$ where p is the precision of the quadrature formula.

Theorem 5 : Geometric Mean derivative-based Closed Trapezoidal Rule ($n = 1$) with the error term is

$$\begin{aligned} \textstyle\int_a^b f(x)dx \quad &\approx \quad \tfrac{b-a}{2}(f(a) + f(b)) - \tfrac{(b-a)^3}{12} f''(\sqrt{ab}) \\ &\quad -\tfrac{(b-a)^3}{24}(\sqrt{b} - \sqrt{a})^2 f^{(3)}(\xi), \end{aligned} \tag{11}$$

where $\xi \in (a, b)$. This is fourth order accurate with the error term

$$E_1[f] = -\frac{(b-a)^3}{24}(\sqrt{b} - \sqrt{a})^2 f^{(3)}(\xi).$$

Proof : Let

$$f(x) = \frac{x^3}{3!}, \quad \frac{1}{3!}\int_a^b x^3 dx = \frac{1}{24}(b^4 - a^4);$$

$$\frac{b-a}{2}(f(a) + f(b)) - \frac{(b-a)^3}{12} f''(\sqrt{ab}) = \frac{b-a}{3!.2}(b^3 + a^3 - (b-a)^2\sqrt{ab}).$$

Therefore,

$$\frac{1}{24}(b^4 - a^4) - \frac{b-a}{3!.2}(b^3 + a^3 - (b-a)^2\sqrt{ab}) = -\frac{(b-a)^3}{24}(\sqrt{b} - \sqrt{a})^2.$$

Therefore the Error term is,

$$E_1[f] = -\frac{(b-a)^3}{24}(\sqrt{b} - \sqrt{a})^2 f^{(3)}(\xi).$$

Theorem 6 : Geometric Mean derivative-based Closed Simpson's 1/3rd Rule ($n = 2$) with the error term is

$$\begin{aligned}&\textstyle\int_a^b f(x)dx \approx \frac{b-a}{6}\left[(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b))\right] - \frac{(b-a)^5}{2880} f^{(4)}(\sqrt{ab})\\ &\textstyle -\frac{(b-a)^5}{5760}(\sqrt{b} - \sqrt{a})^2 f^{(5)}(\xi),\end{aligned} \tag{12}$$

where $\xi \in (a, b)$. This is sixth order accurate with the error term

$$E_2[f] = -\frac{(b-a)^5}{5760}(\sqrt{b} - \sqrt{a})^2 f^{(5)}(\xi).$$

Proof : Let

$$f(x) = \frac{x^5}{5!}, \quad \frac{1}{5!}\int_a^b x^5 dx = \frac{1}{720}(b^6 - a^6);$$

$$\begin{aligned}&\textstyle\frac{b-a}{6}\left[(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b))\right] - \frac{(b-a)^5}{2880} f^{(4)}(\sqrt{ab})\\ &\textstyle= \frac{b-a}{5!.48}(8a^5 + (a+b)^5 + 8b^5 - 2(b-a)^4\sqrt{ab}).\end{aligned}$$

Therefore,

$$\begin{aligned}&\textstyle\frac{1}{720}(b^6 - a^6) - \frac{b-a}{5!.48}(8a^5 + (a+b)^5 + 8b^5 - 2(b-a)^4\sqrt{ab})\\ &\textstyle= -\frac{(b-a)^5}{5760}(\sqrt{b} - \sqrt{a})^2.\end{aligned}$$

Therefore the Error term is,

$$E_2[f] = -\frac{(b-a)^5}{5760}(\sqrt{b} - \sqrt{a})^2 f^{(5)}(\xi).$$

Theorem 7 : Geometric Mean derivative-based Closed Simpson's 3/8th Rule ($n = 3$) with the error term is

$$\begin{aligned}&\textstyle\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{6480} f^{(4)}(\sqrt{ab})\\ &\textstyle -\frac{(b-a)^5}{12960}(\sqrt{b} - \sqrt{a})^2 f^{(5)}(\xi),\end{aligned} \tag{13}$$

where $\xi \in (a, b)$. This is sixth order accurate with the error term

$$E_3[f] = -\frac{(b-a)^5}{12960}(\sqrt{b} - \sqrt{a})^2 f^{(5)}(\xi).$$

Proof : Let

$$f(x) = \frac{x^5}{5!}, \quad \frac{1}{5!}\int_a^b x^5 dx = \frac{1}{720}(b^6 - a^6);$$

$$\tfrac{b-a}{8}\left[f(a)+3f\left(\tfrac{2a+b}{3}\right)+3f\left(\tfrac{a+2b}{3}\right)+f(b)\right]-\tfrac{(b-a)^5}{6480}f^{(4)}(\sqrt{ab})$$

$$=\tfrac{b-a}{5!.648}(81a^5+(2a+b)^5+(a+2b)^5+81b^5-12(b-a)^4\sqrt{ab}).$$

Therefore,

$$\tfrac{1}{720}(b^6-a^6)-\tfrac{b-a}{5!.548}(81a^5+(2a+b)^5+(a+2b)^5+81b^5-12(b-a)^4\sqrt{ab})$$

$$=-\tfrac{(b-a)^5}{12960}(\sqrt{b}-\sqrt{a})^2.$$

Therefore the Error term is,

$$E_3[f]=-\frac{(b-a)^5}{12960}(\sqrt{b}-\sqrt{a})^2f^{(5)}(\xi).$$

Theorem 8 : Geometric Mean derivative-based Closed Boole's rule ($n=4$) with the error term is

$$\begin{aligned}\textstyle\int_a^b f(x)dx \quad &\approx \quad \tfrac{b-a}{90}\left[7f(a)+32f\left(\tfrac{3a+b}{4}\right)+12f\left(\tfrac{a+b}{2}\right)+32f\left(\tfrac{a+3b}{4}\right)+7f(b)\right]\\ &\quad -\tfrac{(b-a)^7}{1935360}f^{(6)}(\sqrt{ab})-\tfrac{(b-a)^7}{3870720}(\sqrt{b}-\sqrt{a})^2f^{(7)}(\xi),\end{aligned} \tag{14}$$

where $\xi\in(a,b)$. This is eighth order accurate with the error term

$$E_4[f]=-\frac{(b-a)^7}{3870720}(\sqrt{b}-\sqrt{a})^2f^{(7)}(\xi).$$

Proof : Let

$$f(x)=\frac{x^7}{7!},\quad \frac{1}{7!}\int_a^b x^7dx=\frac{1}{40320}(b^6-a^6);$$

$$\tfrac{b-a}{90}\left[7f(a)+32f\left(\tfrac{3a+b}{4}\right)+12f\left(\tfrac{a+b}{2}\right)+32f\left(\tfrac{a+3b}{4}\right)+7f(b)\right]-\tfrac{(b-a)^7}{1935360}f^{(6)}(\sqrt{ab})$$

$$=\tfrac{b-a}{7!.768}(97a^7+91a^6b+\cdots+105a^5b^2+91ab^6+91b^7$$

$$+105a^2b^5+91ab^7-2(b-a)^4\sqrt{ab})$$

Therefore,

$$\tfrac{1}{720}(b^6-a^6)-\tfrac{b-a}{7!.768}(97a^7+91a^6b+105a^5b^2+91a^4b^3+91a^3b^4$$

$$+105a^2b^5+91ab^6+97b^7-2(b-a^4\sqrt{ab})$$

$$=-\tfrac{(b-a)^7}{3870720}(\sqrt{b}-\sqrt{a})^2.$$

Therefore the Error term is,

$$E_4[f]=-\frac{(b-a)^7}{3870720}(\sqrt{b}-\sqrt{a})^2f^{(7)}(\xi).$$

The summary of Precision, the orders and the error terms for Geometric mean derivative based Closed Newton- Cotes Quadrature are shown in Table 1.

Table 1 : Comparison of Error terms

Rules	**Precision**	**Order**	**Error terms**
Trapezoidal rule ($n = 1$)	2	4	$-\frac{(b-a)^3}{24}(\sqrt{b}-\sqrt{a})^2 f^{(3)}(\xi)$
Simpson's 1/3rd rule ($n = 2$)	4	6	$-\frac{(b-a)^5}{5760}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi)$
Simpson's 3/8th rule ($n = 3$)	4	6	$-\frac{(b-a)^5}{12960}(\sqrt{b}-\sqrt{a})^2 f^{(5)}(\xi)$
Boole's rule ($n = 4$)	6	8	$-\frac{(b-a)^7}{3870720}(\sqrt{b}-\sqrt{a})^2 f^{(7)}(\xi)$

4. Numerical Results

In this section, to demonstrate the accuracy of the newly derived closed Geometric mean derivative based formula the values of $\int_1^2 e^x dx$ and $\int_1^2 \frac{dx}{1+x}$ are calculated. The precision of Newton-Cotes quadrature and the Geometric Mean derivative based closed Newton Cotes formulas are compared in Table 2 and 3.

We know that

$$\text{Error} = |\text{Exact value} - \text{Approximate value}|$$

Example 1 : Solve $\int_a^2 e^x dx$ and compare the solutions with the CNC and GMDCNC rules.

Solution : Exact value of $\int_1^2 e^x dx = 4.67077427$.

Table 2: Comparison of CNC and GMDCNC rules

Value of n	CNC		GMDCNC	
	Approximate value	Error	Approximate value	Error
$n = 1$	5.053668964	0.382894694	4.710898099	0.040123829
$n = 2$	4.672349035	0.001574765	4.670920823	0.000146553
$n = 3$	4.671476470	0.000702200	4.670841709	0.000067439
$n = 4$	4.670776607	0.000002337	4.670774481	0.000000211

Example 2 : Solve $\int_1^2 \frac{dx}{1+x}$ and compare the solutions with the CNC and GMDCNC rules.

Solution : Exact value of $\int_1^2 \frac{dx}{1+x} = 0.405465108$.

Table 3 : Comparison of CNC and GMDCNC rules

Value of n	CNC		GMDCNC	
	Approximate value	Error	Approximate value	Error
$n = 1$	0.416666667	0.011201559	0.404822031	0.000643077
$n = 2$	0.405555556	0.000090448	0.405453939	0.000011169
$n = 3$	0.405505952	0.000040844	0.405460791	0.000004317
$n = 4$	0.405465768	0.000000660	0.405464989	0.000000119

5. Conclusion

In this paper, a new scheme of Geometric mean derivative-based Closed Newton-Cotes quadrature formulas were presented by using the concept of precision. The main idea behind our approach is that the value of Geometric mean was considered at the derivative. The error bounds for the new quadrature formula were derived by using the difference between the quadrature formula for the monomials and the exact results. Finally, the efficiency of the proposed research is illustrated by an example.

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