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SOME PROPERTIES OF GÂTEAUX AND FRÉCHET DERIVATIVES IN 2-NORMED SPACE

K. SANGITA DEVI
P.G. Department of Mathematics,
D.M. College of Science, Imphal,
Manipur-795001, India

Abstract

It is proved that in a 2- normed space Gâteaux derivative is not necessarily a linear operator. But, it is also proved that the linearity and continuity of Gâteaux derivative do not necessarily imply the linearity of the operator. Further a relation between Fréchet and Gâteaux derivatives is given.

1. Introduction

The concept of differentiation can be introduced in case of operators defined on a normed space into another normed space. For a mapping between two normed spaces, the definition of derivatives was due to R. Gâteaux and M. Fréchet, J. Hadmard and so on. Considering 2-normed space which was introduced in series of paper [2, 5], K. Iséki [3] defined the differentiation of a mapping defined between two 2-normed spaces X and Y in sense of Gâteaux and Fréchet.

Key Words : *2-normed space, Gateaux derivative and Frechet derivative.*

AMS Subject Classification : 56B.

2. Preliminaries

Definition 1 (S. G*ä*hler [2]) : Let X be a linear space over reals of dimension greater than one and let $\|\cdot,\cdot\|$ be a real valued function on $X \times X$ satisfying the following properties

(N1) $\|a, b\| = 0$ if and only if a and b are linearly dependent;

(N2) $\|a, b\| = \|b, a\|$ for every $a, b \in X$;

(N3) $\|a, \alpha b\| = |\alpha|\|a, b\|$, where α is a real and

(N4) $\|a, b + c\| \leq \|a, b\| + \|a, c\|$ for every $a, b, c \in X$.

Then $\|\cdot,\cdot\|$ is called a 2-norm and the pair $(X, \| \cdot \cdot\|)$, a **2-normed space**.

Definition 2 (K. Is*é*ki [3]) : Let $f : X \to Y$ be mapping from$X$ to Y, where X and Y are linear 2-normed spaces. If for an element $x_0 \in X$ there is a mapping of $\delta f(x_0, h) : X \to Y$ satisfying

$$\lim_{t\to 0} \left\| \frac{f(x_0 + th) - f(x_0)}{t} - \delta f(x_0, h), y \right\| = 0$$

for $y \in X$ and $h \in X$ then $\delta f(x_0, h)$ is called the **Gâteaux derivative (variation)** of f at the point x_0. It is also denoted by $\delta f(x_0, h)$ or $f'(x_0)h$ or $f'(x_0)h$ or simply $f'(x_0)$. If the dimension of X is greater than or equal to 2, Gâteaux derivative (variation) is unique. For any $\alpha, \delta f(x_0, \alpha h) = \alpha\delta f(x_0, h)$.

Definition 3 (K. Is*é*ki [2]) : Let X and Y be 2-normed spaces. If for an element $x_0 \in X$, there is a linear mapping $F'(x_0) : X \to Y$ such that for every $h \in X$, we have

$$F(x_0 + h) - F(x_0) = F'(x_0) + r(x_0, h)$$

where

$$\lim_{h\to 0} \frac{\|r(x_0, h), y\|}{\|(h, y)\|} = 0 \quad \text{with} \quad \|(h, y)\| \neq 0,$$

then $F'(x_0)$ is called the Fréchet differentiation at the point x_0.

3. Main Results

We shall give the following properties of Gâteaux and Fr*é*chet derivatives.

Property 1 : Gâteaux derivatve is not necessarily a linear operator.

Proof : Here, $f'(x)$ denotes Gâteaux derivative of mapping f from 2-normed space X to 2-normed space Y. For $y, h \in X$, consider $x_1, x_2 \in X$ and two scalars α and β.

Now,

$$\begin{aligned} \alpha f'(x_1) + \beta f'(x_2) &= \alpha \lim_{t\to 0} \left\| \frac{f(x_1+th) - f(x_1)}{t}, y \right\| + \beta \lim_{t\to 0} \left\| \frac{f(x_2+th) - f(x_2)}{t}, y \right\| \\ &\geq \lim_{t\to 0} \left\| \frac{\{\alpha f(x_1+th) + \beta f(x_2+th)\} - \{\alpha f(x_1) + \beta f(x_2)\}}{t}, y \right\| \\ &\neq \lim_{t\to 0} \left\| \frac{f(\alpha x_1 + \beta x_2 + th) - f(\alpha x_1 + \beta x_2)}{t}, y \right\| \\ &= f'(\alpha x_1 + \beta x_2). \end{aligned}$$

Therefore, Gâteaux derivative is not linear.

Property 2 : If the mapping f is linear then $f'(x_0) = f(h)$.

Proof : Here, f is linear. Then, by the definition

$$\begin{aligned} &\lim_{t\to 0} \left\| \frac{f(x_0+th) - f(x_0)}{t} - f'(x_0), y \right\| = 0 \\ &\Rightarrow \|f(h) - f'(x_0), y\| = 0. \end{aligned}$$

That is, $f'(x_0) = f(h)$.

Property 3 : Linearity and continuity of Gâteaux derivative do not necessarily imply the linearity of the operator f.

Proof : Let $f'(x)$ is linear and continuous. Then,

$$\begin{aligned} & f'(\alpha x_1 + \beta x_2) = \alpha f'(x_1) + \beta f'(x_2) \\ &\qquad = f'(\alpha x_1) + f'(\beta x_2) \\ &\Rightarrow \lim_{t\to 0} \left\| \frac{f(\alpha x_1 + \beta x_2 + th) - f(\alpha x_1 + \beta x_2)}{t}, y \right\| \\ &\qquad \geq \lim_{t\to 0} \left\| \frac{f(\alpha x_1 + th) + f(\beta x_2 + th) - f(\alpha x_1) - f(\beta x_2)}{t}, y \right\| \\ &\Rightarrow f(\alpha x_1 + \beta x_2 + th) - f(\alpha x_1) + \beta x_2) \\ &\qquad \geq f(\alpha x_1 + th) + f(\beta x_2 + th) - f(\alpha x_1) - f(\beta x_2) \\ &\Rightarrow f(\alpha x_1 + \beta x_2) \neq \alpha f(x_1) + \beta f(x_2). \end{aligned}$$

Thus, linearity and continuity of Gâteaux derivative do not necessarily imply the linearity of the operator f.

Property 4 : Fréchet derivative implies Gateaux derivative whereas Gâteaux derivative does not imply Fréchet derivative.

Proof : By the definition of the Fr*é*chet derivative,

$$\lim_{h\to 0}\|F(x+h)-F(x)-F'(x)h,y\|=0$$
$$\Rightarrow \lim_{h\to 0}\left\|\frac{F(x+h)-F(x)}{h}-F'(x),y\right\|=0$$
$$\Rightarrow F' \text{ is Gâteaux differentiable.}$$

Next, we show that Gâteaux derivative does not imply Fr*é*chet derivative of mapping f.

By the definition of Gâteaux derivative,

$$\begin{aligned} f'(x) &= \lim_{t\to 0}\left\|\frac{f(x+th)-f(x)}{t},y\right\| \\ &= \left|\frac{1}{t}\right|\lim_{t\to 0}\|f(x+th)-f(),y\| \\ &\neq \lim_{h\to 0}\frac{\|f(x+th)-f(x),y\|}{\|h,y\|} \\ &= \text{Fréchet derivative of } f. \end{aligned}$$

Thus, the result is obtained.

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