International J. of Pure & Engg. Mathematics (IJPEM)
ISSN 2348-3881, Vol. 4 No. I (April, 2016), pp. 19-30

DISTANCE COPRIME GRAPHS

R. SUGANYA[1] AND K. NAGARAJAN[2]
[1] Research Scholar, Department of Mathematics,
Sri S. R. N. M. College, Sattur-626203, Tamil Nadu, India
E-mail: slrsuganya@gmail.com
[2] Head & Associate professor,
Department of Mathematics,
Sri S. R. N. M College, Sattur-626203, Tamil Nadu, India
E-mail: k_nagarajan_srnmc@yahoo.co.in

Abstract

Let $G = (V, E)$ be a (p, q) graph. A shortest path P is called *distance coprime path* if $(l(P),\ q) = 1$. A *distance coprime graph* $P(G)$ *of graph* $G = (V, E)$ has the vertex set $V = V(G)$ and two vertices in $P(G)$ are adjacent if they have distance coprime path in G. In this paper, we found the distance coprime graph of standard graphs such as path, cycle, wheel, star and complete bipartite graph etc. Also we found that some properties and characterizations for distance coprime graphs.

1. Introduction

By a graph, we mean a finite undirected, connected graph without loops and multiple edges, for terms not defined here, we refer to [7,5]. Distance between, the two vertices in a graph is the length of the shortest path between them [4]. Graph theory [3, 7] has a strong communication with number theory [6. 8]. Using the coprime concept in number

---

Key Words : *Distance, Distance coprime path, Distance coprime graph.*

AMS Subject Classification : 05C12, 05C76.

theory [6. 8]. Using the coprime concept in number theory, we introduce a new concept called distance coprime graph. In this paper, we find the distance coprime graphs of some standard graphs such as path, cycle, wheel, star, complete graph, complete bipartite graph etc. Also it is found that some properties and characterizations for distance coprime graphs.

We know that line graphs, total graph, antipodal graphs [2], radial graphs, distant divisor graphs [9], etc are some graph theoretic functions. Likewise, distance coprime graph is a new graph theoretic function. First we give some concepts in number theory.

Definition 1.1 [8] : Let a and b be two positive integers. If a and b are said to be coprime if $gcd(a, b) = 1$. It is denoted by $(a, b) = 1$.

Next we give the definitions of some Number theoretic functions.

Definition 1.2 [6] : Given a positive integer n, let ϕ(n) denote the number of positive integers not exceeding n that are coprime or relatively prime to n.

In this paper, ϕ denotes the number of integers which are coprime with q, where q is the number of edges of a graph. We use the term "coprimes of q" means that the positive integers, which are coprime with q.

Example 1.3 : Consider $n = 9$, it has the coprimes $1, 2, 4, 5, 7, 8$, we find that $\phi(9) = 6$.

Definition 1.4 [6] : Given a positive tive integer n, let ψ(n) denote the sum of coprimes of n.

In this paper, ψ denote the sum of coprimes of q.

Example 1.5 : $\psi(9) = 1 + 2 + 4 + 5 + 7 + 8 = 27$.

Notation 1.6 : Let G be a (p, q)- graph,

(i) $c_1, c_2, \ldots, c_\phi$ denote the coprimes of q with $c_1 = 1$ and $c_\phi = q - 1$ and $c_1 < c_2 <, \ldots, < c_\phi$.

(ii) $d_1, d_2, \ldots, d_p$ denote degrees of vertices $v_1, v_2, \ldots, v_p$ of a graph G respectively.

(iii) For any real number x, $\lfloor x \rfloor$ denotes the greatest positive integer $\leq x$.

(iv) The length of the path P denoted by $l(P)$which is the number of edges in the path.

Result 1.7 [1] :

(i) ϕ is always even

(ii) $\phi(p) = p - 1$. if p is prime

(iii) If the integer $n > 1$ has the prime factorization $n = p_1^{k_1}.p_2^{k_2} \ldots, p_r^{k_r}$ $k_i \geq 1$ we have $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_r})$.

Result 1.8 : $\psi(n) = \frac{1}{2}\ n\ \phi(n)$

For $n = 9$, $\psi(n) = 27$.

Result 1.9 [6] : $\phi(n) = \frac{n}{2}$ if and only if $n = 2^k$ for some $k \geq 1$.

Proof : Let $n = 2^k$ for some $k \geq 1$, $\phi(n) = \phi(2^k)= 2^k(1 - \frac{1}{2})= \frac{n}{2}$.

Conversely let, $\phi(n) = \frac{n}{2}$, suppose $n = 2^k N$ where N is odd.

Then $\phi(n)= \phi(2^k)\phi(N) \Rightarrow \frac{n}{2} = \frac{2^k}{2}\phi(N) \Rightarrow n = 2^k\phi(N)$

Thus $\phi(N) = N$, hence $N = 1$. Therefore $n = 2^k$ for some $k \geq 1$. $\square$

2. Main Results

In this section we define distance coprime path and distance coprime graph. Also we find the distance coprime graph of standard graphs. The concept of distant divisor graph was established in [9]. It is defined as follows,

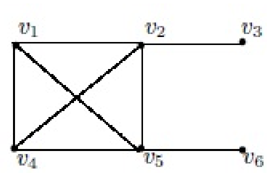
Definition 2.1 [9] : Let $G = (V, E)$ be a (p, q)-graph. A shortest path P is called distant divisor path if its length $l(P)$ divides q.

Definition 2.2 [9] : Let $G = (V, E)$ be a (p, q)-graph. The distant divisor graph $D(G)$ of a graph G has the vertex set $V = V(G)$ and two vertices in $D(G)$ are adjacent if they have distant divisor path in G.

Motivated by this definition we define distance coprime path and distance coprime graph as follows.

Definition 2.3 : Let $G = (V, E)$ be a (p, q)-graph. A shortest path P is called distance coprime path if $(l(P), q) = 1$.

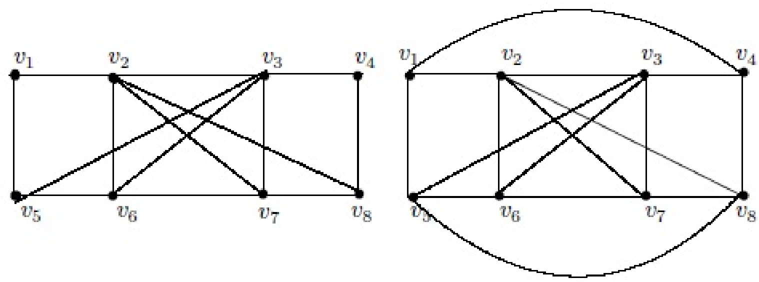
Example 2.4 : Consider the graph



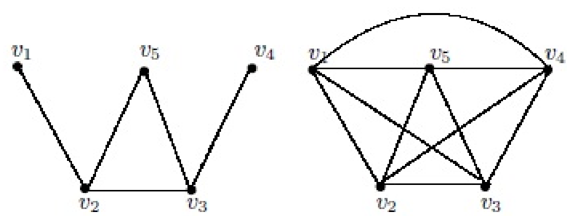
Here $q = 8$, clearly the path $v_3v_2v_5v_6$ is a distance coprime path.

Defintion 2.5 : Let $G = (V, E)$ be a (p, q)-graph. The distance coprime graph $P(G)$ of a graph G has the vertex set $V = V(G)$ and two vertices in $P(G)$ are adjacent if there exists a distance coprime path between them in G.

Example 2.6 : The graph G and its distance coprime graph $P(G)$ are shown below.



G $P(G)$



G $P(G)$

Observation 2.7 : Since every edge in a graph G is a distance coprime path, G is isomorphic to a spanning subgraph of $P(G)$.

Theorem 2.8 : If G is complete, then $P(G) \cong G$.

Proof : Since G is complete, $d(v_i, v_j) = 1$ for all i, j and $i \neq j$.

From the Theorem 2.7 the only distance coprime paths are edges of G and hence $P(G) \cong G$. □

Theorem 2.9 : Distance coprime graph is not a complement of distant divisor graph.

Proof : For any q, we observe that 1 is both divisor as well as a coprime. Hence every edge in G can be considered as both distant divisor path and distance coprime path. Thus distant divisor graphs and distance coprime graphs never complements to each other. □

Next, we will see the distance coprime graph of a graph with prime size.

Theorem 2.10 : Let G be a graph of prime size q, then

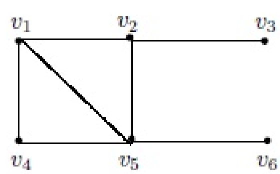
(i) $P(G) \cong K_p$ if G is not a path.

(ii) $P(G) \cong K_p - e$ if G is path and e is the edge joining the end vertices of the path.

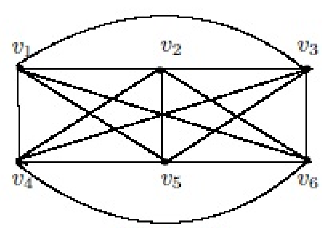
Proof Let G be a graph of prime size q.
Case (i) : Suppose G is not a path. Since q is prime, the coprimes of q are 1, 2, 3, 4 ... $q-1$.
Since G is not a path, the shortest distance between any two vertices $\leq q-1$, and so all pair of vertices in $P(G)$ are adjacant, $P(G) \cong K_p$.
Case (ii): Suppose $G \cong P_p$.
Since G is a path, there is no distance coprime path of length q, so the end vertices are not adjacant. The shortest distance between any two vertices must be $\leq q-1$. So all the pair of vertices of P_p are adjacent except the end vertices of P_p.
From observation 2.7, it follows that $P(G) \cong K_p - e$. □

Example 2.11 : Consider the following graphs of prime size.

(i) Here $q = 7$

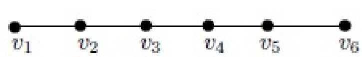


G

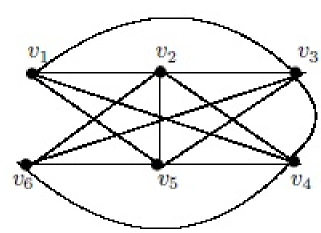


$P(G) = K_6$

Here $q = 5$



G



$P(G) = K_6 - e$

Now we will see the distance coprime graph of graph of diameter 2.

Theorem 2.12 : Let G be a graph with p vertices and q edges. If $\text{diam}(G) = 2$, then

$$P(G) \cong \begin{cases} K_p & \text{if q is odd} \\ \\ G & \text{if q is even} \end{cases}$$

Proof : Let G be a graph with p vertices and q edges and $\text{diam}(G) = 2$.

Case (i) : Suppose q is odd.

Since diam $G = 2$, $d(u, v) = 1$ or 2 for all $u, v \in V(G)$.

Since q is odd, 2 is coprime of q. Clearly $P(G) \cong K_p$.

Case (ii): Suppose q is even.

Since q is even and diam $G = 2$, but 2 is not coprime of q, So distance coprime paths are the edges of G only. Hence $P(G) \cong G$. □

Using this result, we can find distance coprime graph of complete bipartite graph, star, and wheel.

Corollary 2.13 : For a complete bipartite graph $K_{m,n}$.

$$P(K_{m,n}) \cong \begin{cases} K_{m,n} & \text{if } mn \text{ is even} \\ \\ K_{m+n} & \text{if } mn \text{ is odd} \end{cases}$$

Proof : Since $\text{diam}(K_{m,n}) = 2$ and size of $K_{m,n}$ is mn, by the Theorem 2.12, the result follows. □

Corollary 2.14 : For a star $K_{1,p}$ $P(K_{1,p}) \cong \begin{cases} K_{1+p} & \text{if } p \text{ is odd} \\ \\ K_{1,p} & \text{if } p \text{ is even} \end{cases}$

Corollary 2.15 : For a wheel $W_p, P(W_p) \cong K_p$.

Proof : Since $\text{diam}(W_p) = 2$ and size of W_p is even, by the Theorem 2.12 it follows that $P(W_p) \cong K_p$. □

Next we find the characterization for the graph and its distance coprime graph are isomorphic.

Theorem 2.16 : Let G be a (p, q)-graph and q a composite number, Then $P(G) \cong G$ if and only if $\text{diam}(G) \leq c_2 - 1$, where c_2 is the second coprime of q.

Proof : Let $P(G) \cong G$. Suppose $\text{diam}(G) \geq c_2 > 1$. Then there is a distance coprime path of length c_2 in G and the size of $P(G)$ will exceed the size of G, which is a contradiction to $P(G) \cong G$. Hence $\text{diam}(G) \leq c_2 - 1$.

Conversely, Suppose diam $(G) \leq c_2-1$. Then there is no distance coprime path of length c_2. Thus the distance coprime paths in $P(G)$ are edges only, and clearly $P(G) \cong G$. □

Now we find the property of distance coprime graph of a tree.

Theorem 2.17 : If G is a tree, then P(G) is either isomorphic to G or contains a cycle.

Proof : Let G be a tree. Suppose edges are the only distance coprime paths of G, then clearly $P(G) \cong G$.

Suppose G has distance coprime path other than edges, then in G we join the two vertices of that distance coprime path which produces a cycle in $P(G)$. □

Next we have to discuss about the distance coprime graph of cycle.

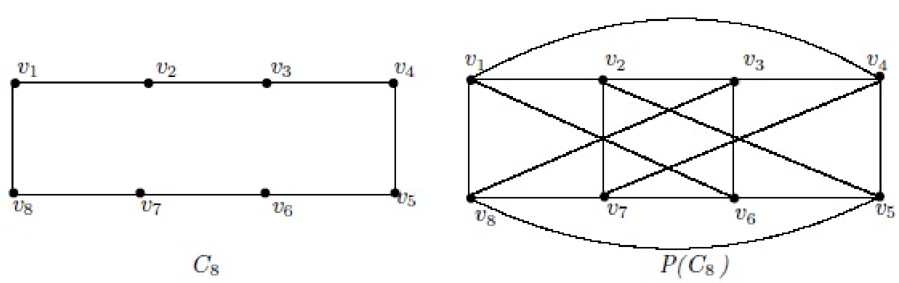
Theorem 2.18 : For a cycle C_p, $P(C_p) \cong \begin{cases} K_p & \text{if } p \text{ is prime} \\ \\ \phi\text{ -regular} & \text{if } p \text{ is composite} \end{cases}$

Proof : Let C_p be a cycle with prime number of edges. Then by Theorem 2.10, $P(C_p) \cong K_p$.

Suppose p is not prime, let $c_1, c_2 \dots c_{\frac{\phi}{2}} \dots c_\phi$ be the coprimes of p.

Since cycle is 2-connected , $diam(C_p) \leq \lfloor \frac{p}{2} \rfloor$. We observe that $c_{\frac{\phi}{2}} \leq \lfloor \frac{p}{2} \rfloor$. For each $c_i \leq \lfloor \frac{p}{2} \rfloor$ there are two paths of length c_i in C_p, from every vertex and there by two edges in $P(C_p)$ contributing degree 2 to each vertex. Thus the degree of each vertex in $P(C_p)$ is $2\frac{\phi(p)}{2}$ regular. Hence $P(C_p)$ is ϕ-regular. □

Example 2.19 : For C_6, we see that $P(C_6) \cong C_6$ which is $2-regular$ and note that $\phi = 2$.



Here $p = q = 8$ and $\phi = 4$, $P(C_8)$ is 4 - regular.

Corollary 2.20 : For a cycle C_p, $q(P(C_p)) = \begin{cases} \frac{p(p-1)}{2} & \text{if } p \text{ is prime} \\ p\phi & \text{if } p \text{ is composite} \end{cases}$

Proof: Suppose p is prime, $P(C_p) \cong K_p$. Thus $q(P(C_p)) = \frac{p(p-1)}{2}$.

Suppose p is composite, $P(C_p)$ is $\phi-$ regular. $2q(P(C_p)) = \sum d_i = p\phi$ and so,

$q(P(C_p)) = \frac{p}{2}\phi$. □

Definition 2.21 [**4**] : The k^{th} power G^k of a graph G is a graph that has the same set of vertices $V(G)$ and two vertices are adjacent when their distance in G is at most k. G^2 is the square of G and G^3 is the cube of G.

Definition 2.22 [**9**] : Two vertices of a (p, q) - graph G are said to be $k - chordal$ to each other if the distance between them is equal to k . The $k - chordal$ graph of a graph G has the vertex set $V = V(G)$ and two vertices are said to be adjacent if they are $k - chordal$ in G. It is denoted by $C_k(G)$

Definition 2.23 [**9**] : The $(k_1, k_2, k_3, ..., k_r)-$ chordal graph of a (p, q) - graph G has the vertex set $V = V(G)$ and two vertices are said to be adjacent, if the distance between them is k_i $(i = 1, 2, 3, \ldots r)$ and $k_1 < k_2 < \cdots < k_r$ and it is denoted by $C_{(k_1,k_2,\ldots k_r)}(G)$.

Observation 2.24 [**9**] : Let G be a (p, q) - graph. Then
(i) If $k_1, k_2, \ldots k_r$ are all divisors of q, then $C_{(k_1,k_2,\ldots k_r)}(G) \cong D(G)$
(ii) $C_{(1,2,\ldots k)}(G) \cong G^k$.

Observation 2.25 : Let G be a $(p, q)-$graph. Then
(i) If $c_1, c_2, \ldots c_\phi$ are all coprimes of q, then $C_{(c_1,c_2,\ldots c_\phi)}(G) \cong P(G)$.

Next we have to see that distance co-prime graph of a graph G of diam$(G) = 3$.

Theorem 2.26 : If diam$(G) = 3$, then

$$P(G) \cong \begin{cases} G & \text{if } q \text{ even and } q \equiv 0 \ (mod 6) \\ C_{1,3}(G) & \text{if } q \text{ even and } q \neq 0 \ (mod 6) \\ G^2 & \text{if } q \text{ odd and } q \equiv 0 \ (mod 3) \\ K_p & \text{if } q \text{ odd and} q \neq 0 \ (mod 3) \end{cases}$$

Case (i) : $q \equiv 0$ (mod 6), and q is even.
Here q is even and $2, 3, 4$ are not coprimes of q. Here $c_1 = 1$ and $c_2 = 5$.
Since diam$(G) = 3$, $d(u, v) = 1, 2, 3 \forall\ u, v \in V(G)$. Thus diam $(G) < c_2 - 1$
From the Theorem 2.16, it follows that $P(G) \cong G$.
Case (ii) : $q \neq 0$ (mod 6) and q is even.
Since q is even, 2 is not a coprime of q. Here $c_1 = 1$, $c_2 = 3$.

Since diam$(G) = 3$, $d(u, v) = 1, 2, 3 \forall u, v \in V(G)$. Then there exists a distance coprime path of length 3. Thus $P(G) \cong C_{1,3}(G)$.

Case (iii) : q is odd and $q \equiv 0 \pmod 3$, Here $c_1 = 1$, $c_2 = 2$.

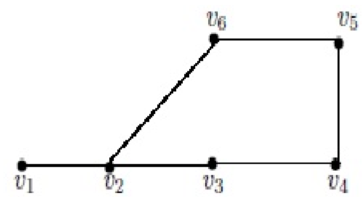
Also 3 is not coprime with q. Thus $P(G) \cong G^2$.

Case (iv) : q is odd and $q \neq 0 \pmod 3$.

Then 2 and 3 are coprimes of q, and $c_1 = 1, c_2 = 2, c_3 = 3$. and $\text{diam}(G) = 3$

Thus each pair of vertices in $P(G)$ are adjacent. Hence $P(G) \cong K_p$. □

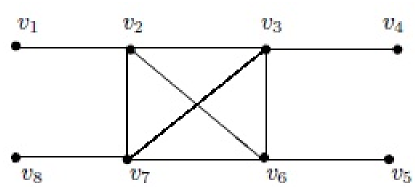
Example 27 : Consider the graph G



Here $q = 6$, $q \equiv 0 \pmod 6$ and q is even. $\text{diam}(G) = 3 \leq c_2 - 1$.

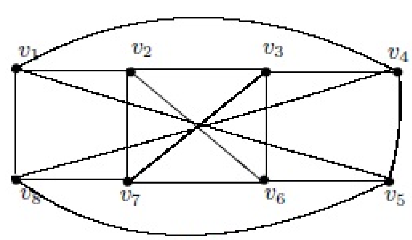
Thus $P(G) \cong$ G.

Consider the graph G,



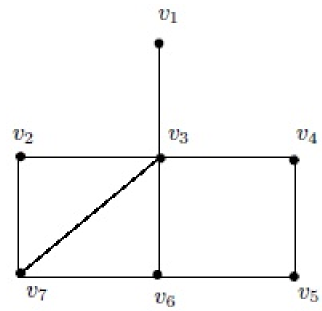
Here $q = 10$ and $c_1 = 1, c_2 = 3$ and $\text{diam}(G) = 3$.

The distance co-prime graph of G is given below. Here $P(G) \cong C_{1,3}(G)$.



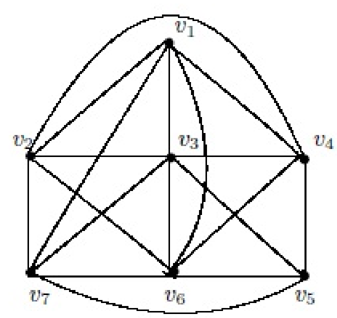
P(G)

Now , Consider the graph G,



Here $q = 9$, $c_1 = 1, c_2 = 2$ and $\text{diam}(G) = 3$.

The distance co-prime graph of G is given below, Here $P(G) \cong G^2$.



Theorem 2.28 : There exists no graph G $(p > 2)$ such that $P(G) \cong P_p$.

Proof : Suppose there exists a graph G $(p > 2)$ such that $P(G) \cong P_p$. From the Observation 2.7, G must be a spanning subgraph of P_p. The spanning subgraph of P_p is P_p only. So G must be P_p and $P(P_p) \ncong P_p$ for $p > 2$. Thus there does not exists G such that $P(G) \cong P_p$.

Theorem 2.29 : Let G be a (p, q) graph. Then $P(G) \cong C_p$ if and only if G is either even cycle of diam ≤ 3 or $G \cong C_3$.

Proof : Suppose G is a even cycle of diam≤ 3, then $q \leq 6$.

Since $\text{diam}(G) \leq c_2 - 1$, where c_2 is the second coprime of q, by Theorem 2.16 $P(G) \cong G$.

Thus $P(G) \cong C_p$.

Suppose $G \cong C_3$ then clearly $P(G) \cong C_3$.

Conversely , Suppose $P(G) \cong C_p$, from Observation 2.7 G must be a spanning subgraph C_p. The only spanning subgraphs of C_p are P_p and C_p, but
$P(P_p) \ncong C_p$, and we note that $P(C_p) \cong C_p$ only if G is a even cycle of diam ≤ 3 or $G \cong C_3$. □

Theorem 2.30 : If G be a bipartite graph with 2^k edges then $P(G)$ is isomorphic to complete bipartite graph.

Proof : Let G be a bipartite graph with $q = 2^k$ edges. Since the coprimes of q are all odd numbers $< q$. Therefore all distance coprime paths in $P(G)$ are at of odd distance and that makes even cycles only, so $P(G)$ is bipartite . We choose an arbitrary vertex u and define a partition (X, Y) of $V(G)$ in $P(G)$ by setting

$$X = \{x \in \ V(G)/d(u, x) = even\}$$

$$Y = \{y \in V(G)/d(u, y) = odd\}$$

We shall show that (X, Y) is a bipartition of $P(G)$. Suppose that v and w are two vertices of X. Let P be the shortest $(u, v) - path$ and Q be the shortest $(u, w) - path$. Denote by u_1 the vertex common to P and Q. Since P and Q are shortest paths, the $(u, u_1) - sections$ of both P and Q are shortest $(u, u_1) - paths$ and, therefore have the same length. Now since the lengths of the $(u_1, v) - section$ P_1 of P and the $(u_1, w) - section$ Q_1 of Q must have the same parity. It follows that the (v, w) path $P_1^{-1}Q_1$ is of even length, if v were joined to w, $P_1^{-1}wv$ would be cycle of odd length , contrary to the hypothesis.

Therefore no two vertices in X are adjacent; similarly, no two vertices in Y are adjacent. Hence (X, Y) is a bipartition of $P(G)$.

Let $x \in X$ and $y \in \ Y$. Suppose x and y are not adjacent in $P(G)$, then there is no distance coprime path between them, $d(x, y) = even$. Since $x \in X$, $d(u, x) = even$. Then, $d(u, y) = even$ which implies $y \in X$, which is a contradiction. Therefore each $x \in X$ and each $y \in Y$ are adjacent in $P(G)$. Hence $P(G)$ is isomorphic to complete bipartite graph. □

3. Conclusion

Further we are trying to establish the distance coprime graphs of following classes of graphs, edge subdivision of graphs and vertex switching of graphs. Also we are going to check whether the following operations are commutative or not, distance coprime and subdivision, distance coprime and switching.

References

[1] Apostol T. M., Introduction to Analytic Number Theory, third edition, Narosa edition, (1991).

[2] Aravamudhan R. and Rajendran B., A note on antipodal graphs, Discrete Math, 58 (1986), 303-305.

[3] Bondy J. A. and Murty U. S. R., Graph Theory with Applications, North Holland, Fifth Printing, (1983).

[4] Buckley F. and Harary F., Distance in Graphs, Addison-Wesley, Reading, (1990).

[5] Chartrand G. and Ping Zhang, Introduction to Graph Theory, Tata McGraw-Hill Publishing company Limited, New Delhi, (2006).

[6] Burton David M., Elementary Number Theory, Tata McGraw-Hill Publishing Company Limited, (2006).

[7] Harary F., Graph Theory, Addison-Wesley Publishing Company, Reading, Mass, (1972), 14(6) (2004),481-492.

[8] Ivan Niven and Herbert S. Zuckerman, An Introduction to the Theory of Numbers, H. S. Wiley Eastern Limited.

[9] Saravanakumar S. and Nagarajan K., Distant divisor graphs, International Journal of Mathematical Sciences and Engineering Applications, 7(IV) (2013), 83-97.