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ON A CERTAIN SUBCLASS OF NORMALIZED ANALYTIC FUNCTIONS INVOLVING THE LINEAR DIFFERENTIAL OPERATOR

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Abstract

Let A denotes the class of normalized analytical function on open unit disc as given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. We studied two subclasses, $K_p(D_\lambda^m; \gamma, \mu, m, \beta)$ and $\tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$ with some examples. D_λ^m be the linear differential operator operating on given subclasses. Here we discussed coefficient inequality, integral mean, extreme points, closure theorem, growth and distortion theorem for the subclass, $\tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$.

1. Introduction and Preliminaries

Let A denotes class of normalized analytical functions in open unit disk $U = \{z : |z| < 1\}$ given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

For $f \in A$ [1] has introduced following differential operator
$D_\lambda^n : A \to C$ defined by

$$D_\lambda^n(f(z)) = z + \sum_{k=2}^{\infty} \; [(1-\lambda)k^n + \lambda^{n+k-1}_{\;\;n}C]a_k z^k \quad (z \in U). \tag{1.2}$$

Definition 1.1 : A function f in A is said to be close-to-convex in U, of order α, that is, $f \in C(\alpha)$, if and only if

$$Re\{f(z)\} > \alpha \quad (z \in U. \tag{1.3}$$

Definition 1.2 : A function f in A is said to be close-to-star like of order α $(0 \leq \alpha < 1)$ that is $f \in CS^*(\alpha)$ if and only if

$$Re\left\{\frac{f(z)}{z}\right\} > \alpha \quad (z \in U) \tag{1.4}$$

We note that the classes $C(0) = C$, $CS^*(0) = CS^*$ are well known classes of close-to-convex and close-to-star like functions in U.

Definition 1.3 : For two functions f and g analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U and write

$$f(z) \prec g(z) \;\; (z \in U). \tag{1.5}$$

If there exit Schwarz function $w(z)$, analytical in U with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \quad (z \in U). \tag{1.6}$$

J. E. Littlewood has introduced in [7] following subordination theorem which we stated as lemma.

Lemma 1.1 : Let f and g analytic in unit disc and suppose $g \prec f$, then for $0 < t < \infty$

$$\int_0^{2\pi} |g(re^{i\phi})|^t d\theta \leq \int_0^{2\pi} |f(re^{i\phi})|^t d\theta \quad (0 \leq r < 1, t > 0). \tag{1.7}$$

Strict equality hold for $0 \leq r < 1$ unless f is constant or $w(z) = \alpha z, \;\; |\alpha = 1$.

Definition 1.4 :

$$K_p(D_\lambda^m; \gamma, \mu, m, \beta), = \left\{ f \in B : \left| \frac{1}{p\gamma}((p-u)\frac{D_\lambda^m f}{z} + u(D_\lambda^m f)' - p) \right| < \beta \right\} \tag{1.8}$$

where $z \in U$, $\gamma \in C|\{0\}, 0 < \beta \leq 1 \;\; 0 \leq \mu \leq p, \;\; m \in N \cup \{0\}, D_\lambda^m f$ is defined in (1.2).

Example : If $f(z) = z$, then for $\gamma = 1, \mu = p, m = 0, 0 < \beta \leq 1$, show that $f(z), \in K_p(D_\lambda^m; \gamma, \mu, m, \beta)$.

Answer : For $\gamma = 1, \mu = p, m = 0, 0 < \beta \leq 1$,

$$\begin{aligned} \left|\frac{1}{p\gamma}\left((1p-u)\frac{D_\lambda^m f}{z} + u(D_\lambda^m f)' - p\right)\right| &= \left|\frac{1}{p}\left((p-p)\frac{D_\lambda^0 f}{z} + (D_\lambda^0 f)' - 1\right)\right| \\ &= |(D_\lambda^0 f)' - 1| \\ &= |f(z)' - 1| \\ &= 1 - 1| \\ &< \beta. \end{aligned}$$

Hence $f(z) \in K_p(D_\lambda^m, \gamma, \mu, m, \beta)$.

Definition 1.5 :

$$\begin{aligned} \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta) &= \{f \in K_p(D_\lambda^m; \gamma, \mu, m, \beta) : \sum_{k=2}^{\infty}(p + (k-1)\mu) \\ & [(1-\lambda)k^n + \lambda^{n+k-1}_{\ n}C]|a_k| \leq p|\gamma|\beta\} \end{aligned} \tag{1.9}$$

Remark 1 : If $f \in K_p(D_\lambda^m; \gamma, \mu, m, \beta), \gamma = 1, \mu = 0, m = 0$, then $f \in C(1-\beta)$.

Remark 2 : If $K_p(D_\lambda^m; \gamma, \mu, m, \beta), \gamma = 1, \mu = 0, m = 0$, then $f \in CS^*(1-\beta)$.

Example : If $f(z) = \sin z$, then for $\gamma = 1, \mu = p, m = 0, \beta = 1$, show that $f(z) \in K_p(D_\lambda^m; \gamma, \mu, m, \beta)$ iff $|\cos|^2 - 2Re\ \cos z < 0$.

Answer : $|\cos z|^2 - 2Re\ \cos z < 0$

$Cosz.\overline{Cosz} - 2Re\ \cos z < 0$

$Cosz.\overline{Cosz} - (\cos z + \overline{Cosz}) < 0.$

$Cosz.\overline{Cosz} - Cosz - \overline{Cosz} + 1 < 1$

$|\cos z - 1|^2 < 1.$

$|\cos z - 1| < 1$

$$\begin{aligned} \left|\frac{1}{p\gamma}\left((p-u)\frac{D_\lambda^m f}{z} + u(D_\lambda^m f)' - p\right)\right| &= \left|\frac{1}{p}\left((p-p)\frac{D_\lambda^0 f}{z} + (D_\lambda^0 f)' - p\right)\right| \\ &= |(D_\lambda^0 f)' - 1| \\ &= |(\sin z)' - 1| \\ &= |\cos z - 1| \\ &< 1. \end{aligned}$$

Hence $f(z) \in K_p(D_\lambda^m; \gamma, \mu, m, \beta)$.

Example : If $f(z) = \sin z$, then for $\gamma = \sum\limits_{k=2}^{\infty} \frac{1}{(k-1)}, \mu = 0, m = 0,\ \beta = \frac{1}{p'}$. Show that $f(z) = \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$.

Answer:

$$\begin{aligned}
f(z) &= \sin z = z - \frac{z^2}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\
&= z + \sum_{k=2}^{\infty} a_k z^k \qquad \text{where } a_k = 0 \text{ if } k \text{ is even};\ = \tfrac{-1}{(k)!} \text{ if } k \equiv 4 \text{ (mod 3)};\ = \tfrac{1}{(k)!} \text{ if } k \equiv 4 \text{ (mod1)} \\
&= \sum_{k=2}^{\infty} (p + (k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]|a_k| \\
&= \sum_{k=2}^{\infty} (p + (k-1)1)[(1-\lambda)k^0 + \lambda^{0+k-1}{}_0C]|a)k| \\
&= \sum_{k=2}^{\infty} k|a_k| \\
&= \sum_{k=2}^{\infty} k \left| \frac{(-1)^k}{(k)!} \right| \\
&\leq \sum_{k=2}^{\infty} \left| \frac{1}{(k-1)!} \right| \\
&\leq p|\gamma|\beta.
\end{aligned}$$

Hence $f(z) \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$.

2. Coefficient Inequality, Growth and Distortion Theorems, Closure Theorems

Theorem 2.1 : Let $f(z) \in B$ satisfy

$$\sum_{k=2}^{\infty} (p + k - 1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]|a_k| \leq p|\gamma|\beta \tag{2.1}$$

$\gamma \in C|\{0\},\ 0 < \beta \leq 1,\ \ 0 < \mu \leq p,\ \ \lambda \geq 0,\ m \in N \cup \{0\}/$

Then $f \in K_p(D_\lambda^m; \gamma, \mu, m, \beta)$.

Proof Assume (2.1) is valid for $f(z) \in B$ and γ $(\gamma \in C|\{0\}), \beta(0 < \beta \leq 1), \mu$ $(0 < \mu \leq$

$p), m \in N \cup \{0\}$. Using (1.8) we have

$$(p-u)\frac{D_\lambda^m f}{z} + u(D_\lambda^m f)' - p = \frac{(p-u)}{z}\left[z + \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_k z^k\right]$$

$$+\mu\left[1 + \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]ka_k z^{k-1}\right] - p$$

$$= (p-u)\left[1 + \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_k z^{k-1}\right]$$

$$+\mu\left[1 + \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]ka_k z^{k-1} - p\right]$$

$$= \sum_{k=2}^{\infty}(p-u)[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_k z^{k-1}$$

$$+ \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]k\mu a_k z^{k-1}$$

$$= \sum_{k=2}^{\infty}(p+\mu(k-1)[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_k z^{k-1}$$

$$\begin{aligned}\left|(p-u)\frac{D_\lambda^m f}{z} + u(D_\lambda^m f)' - p\right| &\leq \sum_{k=2}^{\infty}(p+\mu(k-1)[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]|a_k||z^{k-1}| \\ &\leq \sum_{k=2}^{\infty}(p+\mu(k-1)[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]|a_k| \\ &\leq p|\gamma|\beta.\end{aligned}$$

Therefoer $f \in K_p(D_\lambda^\mu; \gamma, \mu, m, \beta)$.

Corollary 2.1 : If $f(z) \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$ then $f(z) \in K_p(D_\lambda^m; \gamma, \mu, m, \beta)$.

Theorem 2.2 : If $f(z) \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$ then

$$|a_k| \leq \frac{p|\gamma|\beta}{[p+\mu(k-1)][(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]} \quad k \geq 2.$$

Proof : Given that $f(z) \in \tilde{K}_p(D_\lambda^m, \gamma, \mu, m, \beta)$. Therefore,

$$\sum_{k=2}^{\infty}[p=\mu(k-1)][(1-\mu)k^n + \lambda^{n+k-1}_{\ \ n}C]|a_k| \leq p|\gamma|\beta$$

$$[p+\mu(k-1)][(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]|a_k| \leq p|\gamma|\beta$$

$$|a_k| \leq \frac{p|\gamma|\beta}{[p+\mu(k-1)][(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]}.$$

Theorem 2.3 : Let function $f(z)$ defined by (1.1) be in class $\tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$ then

$$|z| - \frac{p|\gamma|\beta}{[p+\mu][(1-\lambda)2^n + \lambda n]}|z|^2 \leq |f(z)| \leq |z| + \frac{p|\gamma|\beta}{[p+\mu][(1-\lambda)2^n + \lambda n]}|z|^2. \quad (2.2)$$

Equality is attained for function $f(z)$ given by $f(z) = z + \frac{p|\gamma|\beta}{[p+\beta][(1-\lambda)2^n+\lambda n]} z^2$.

Proof :

$$[p+\nu][(1-\lambda)2^n + \lambda n]\sum_{k=2}^{\infty}|a_k| = (p+\mu)[(1-\lambda)2^n + \lambda^{n+k-1}{}_nC]\sum_{k=2}^{\infty}|a_k|$$
$$\leq \sum_{k=2}^{\infty}(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]|a_k|$$
$$\leq p|\gamma|\beta.$$

Therefore,

$$\sum_{k=2}^{\infty}|a_k| \leq \frac{p|\gamma|\beta}{[p+\mu][(1-\lambda)2^n + \lambda n]}. \quad (2.3)$$

Also $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and using (2.3)

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{k=2}^{\infty}|a_k||z^k| \\ &\leq |z| + |z|^2\sum_{k=2}^{\infty}|a_k| \leq |z| + |z|^2 \tfrac{p|\gamma|\beta}{[p+\mu][(1-\lambda)2^n+\lambda n]} \end{aligned} \quad (2.4)$$

Similarly

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{k=2}^{\infty}|a_k||z^k| \\ &\geq |z| - |z|^2\sum_{k=2}^{\infty}|a_k| \geq |z| - |z|^2 \tfrac{p|\gamma|\beta}{[p+\mu][(1-\lambda)2^n+\lambda n]}. \end{aligned} \quad (2.5)$$

Using (2.4) and (2.5)

$$|z| - \frac{p|\gamma|\beta}{[p+\mu][(1-\lambda)2^n + \lambda n]}|z|^2 \leq |f(z)| \leq |z| + \frac{p|\gamma|\beta}{[p+\mu][(1-\lambda)2^n + \lambda n]}|z|^2.$$

Hence

$$|z| - \frac{p|\gamma|\beta}{[p+\mu][(1-\lambda)2^n + \lambda^{n+1}{}_nC]}|z|^2 \leq |f(z)| \leq |z| + \frac{p|\gamma|\beta}{[p+\mu][(1-\lambda)2^n + \lambda^{n+1}{}_nC]}|z|^2.$$

Equality is attained for function $f(z)$ given by

$$f(z) = \ z + \frac{p|\gamma|\beta}{[p+\mu][(1-\lambda)2^n + \lambda^{n+1}{}_nC]} z^2.$$

Theorem 2.3 : Let function $f(z)$ defined by (1.1) be in class $\tilde{K}_p(D_\lambda^m;\gamma,\mu,m,\beta)$ then

$$1 - \frac{2p|\gamma|\beta}{\mu[(1-\lambda)2^n + \lambda n]}|z| \le |p(z)| \le 1 + \frac{2p|\gamma|\beta}{\mu[(1-\lambda)2^n + \lambda n]}|z|. \tag{2.6}$$

Equality attained for function $f(z)$ given by $f(z) = z + \frac{p|\gamma|\beta}{\mu[(1-\lambda)2^n+\lambda n]} z^2$.

Proof : Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $f'(z) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1}$

$$\begin{aligned} |f(z)| &\le 1 + \sum_{k=2}^{\infty} k|a_k||z^{k-1}| \\ &\le 1 + |z| \sum_{k=2}^{\infty} k|a_k| \end{aligned} \tag{2.7}$$

But

$$\sum_{k=2}^{\infty} [p + \mu(k-1)][(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]|a_k| \le o|\gamma|\beta.$$

Also

$$\begin{aligned} &2p + (k-2)\mu \ge 0 \\ &2p + k\mu - 2\mu \ge 0 \\ &2p + 2k\mu - 2\mu \ge k\mu \\ &\frac{k\mu}{2} \le p + (k-1)\mu. \end{aligned}$$

Similarly

$$\begin{aligned} &[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC](p + (k-1)\mu) \\ &\ge \frac{k\mu}{2}[(1-\lambda)2^n + \lambda^{n+k-1}{}_nC] \sum_{k=2}^{\infty} \frac{k\mu}{2}[(1-\lambda)2^n + \lambda^{n+1}{}_nC]|a_k| \\ &\le \sum_{k=2}^{\infty} (p + (k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]|a_k| \\ &\le p|\gamma|\beta \frac{\mu}{2} \sum_{k=2}^{\infty} k[(1-\lambda)2^n + \lambda n]|a_k| \le p|\gamma|\beta \\ &\sum_{k=2}^{\infty} k|a_k| \le \frac{2p|\gamma|\beta}{\mu[(1-\lambda)2^n + \lambda n]}. \end{aligned}$$

From (2.7)

$$|f(z)| \leq 1 + \frac{2p|\gamma|\beta}{\mu[(1-\lambda)2^n + \lambda n]}|z|. \tag{2.8}$$

Similarly,

$$\begin{aligned} |f(z)| &\geq 1 - \sum_{k=2}^{\infty} k|a_k||z^{k-1}| \\ &\geq 1 - |z| \sum_{k=2}^{\infty} k|a_k| \geq 1 - \tfrac{2[|\gamma|\beta}{\mu[(1-\lambda)2^n+\lambda n]}|z|. \end{aligned} \tag{2.9}$$

(2.8) and (2.9) imnplies that

$$1 - \frac{2p|\gamma|\beta}{\mu[(1-\lambda)2^n + \lambda n]}|z| \leq |f'(z)| \leq 1 + \frac{2p|\gamma|\beta}{\mu[(1-\lambda)2^n + \lambda n]}|z|.$$

Equality attained for function $f(z)$ given by $f(z) = z + \frac{2p|\gamma|\beta}{\mu[(1-\lambda)2^n+\lambda n]}z^2$.

3. Closure Theorem

Theorem 3.1 : Let $f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j}z^k$, $f_j(z) \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$ then for $g(z) = \sum_{j=1}^{l} c_j f_j(z)$. $g(z) \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$, where $\sum_{j=1}^{l} c_j = 1$.

Proof : Let $f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j}z^k$ with $f_j(z) \in \tilde{K}(D_\lambda^m; \gamma, \mu, m, \beta)$.

$$\sum_{k=2}^{\infty}[p + \mu(k-1)][(1-\lambda)2^n + \lambda n]|a_{k,j}| \leq p|\gamma|\beta.$$

$$\begin{aligned} g(z) &= \sum_{j=1}^{l} c_j f_j(z) \\ &= \sum_{j=1}^{l} c_j \left(z + \sum_{k=2}^{\infty} a_{k,j}z^k\right) \\ &= z + \sum_{j=1}^{l} c_j \sum_{k=2}^{\infty} a_{k,j}z^k \\ &= z + \sum_{k=2}^{\infty} z^k \sum_{j=1}^{l} c_j a_{k,j} \\ &= z + \sum_{k=2}^{\infty} e_k z^k \quad \text{where} \quad e_k = \sum_{j=1}^{l} c_j a_{k,j}. \end{aligned}$$

Claim : $g(z) \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$

$$\begin{aligned}
&\sum_{k=2}^{\infty}[p+\mu(k-1)][(1-\lambda)2^n+\lambda n]|e_k| \\
&= \sum_{k=2}^{\infty}[p+\mu(k-1)][(1-\lambda)2^n+\lambda n]\left|\sum_{j=1}^{l} c_j a_{k,j}\right| \\
&\le \sum_{j=1}^{l}\left(c_j\sum_{k=2}^{\infty}[1+\mu(k-1)][(1-\lambda)2^n+\lambda n]|a_{k,j}|\right) \\
&\qquad\qquad \le \sum_{j=1}^{l} c_j|\gamma|\beta \\
&\le p|\gamma|\beta.
\end{aligned}$$

Therefore $g(z) \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$.

4. Extreme Point Theorem

Remark 4.1 : For $\gamma \in C|\{0\}, 0 < \beta \le 1, 0 \le \mu \le p, m \in N \cup \{0\}$ the following functions are in Class $\tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$

$$\begin{aligned}
f_1(z) &= z + \frac{p\beta|\gamma|}{(p+\mu)[(1-\lambda)2^n+\lambda n]}z^2 \quad (z \in U) \\
f_2(z) &= z + \frac{p|\gamma|\beta}{(p+2\mu)[(1-\lambda)3^n+\lambda^{n+2}{}_nC]}z^3 \quad (z \in U) \\
f_3(z) &= z + \frac{z^2}{(p+\mu)[(1-\lambda)2^n+\lambda n]} + \frac{(p|\gamma|\beta-1)}{(p+2\mu)[(1-\lambda)3^n+\lambda^{n+2}{}_nC]}z^3 \quad (z \in U).
\end{aligned}$$

Theorem 4.1 : Let $f_1(z) = z$ and

$$f_k(z) = z + \frac{p|\gamma|\beta}{(p+(k-1)\mu)[(1-\lambda)k^n+\lambda^{n+k-1}{}_nC]}z^k \qquad (k \ge 2). \tag{4.1}$$

Then $f(z) \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$ if and only if $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$ where $\lambda_k \ge 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof : Suppose that

$$\begin{aligned}
f(z) &= \sum_{k=1}^{\infty} \lambda_k f_k(z) \\
&= \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k \left(z + \frac{p|\gamma|\beta}{(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]} z^k \right) \\
&= \left(1 - \sum_{k=2}^{\infty} \lambda_k\right) z + \sum_{k=2}^{\infty} \lambda_k \left(z + \frac{p|\gamma|\beta}{(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]} z^k \right) \\
&= z + \sum_{k=2}^{\infty} \lambda_k \frac{p|\gamma|\beta}{(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]} z^k \\
&= z + \sum_{k=2}^{\infty} a_k z^k \quad \text{where} \quad a_k = \frac{\lambda_k p|\gamma|\beta}{(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]} z^k.
\end{aligned}$$

Claim : $f(z) \in \tilde{K}(D_\lambda^m; \gamma, \mu, m, \beta)$.

$$\begin{aligned}
&\sum_{k=2}^{\infty} [p+\mu(k-1)][(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]|a_k| \\
&= \sum_{k=2}^{\infty} [p+\mu(k-1)][(1-\lambda)k^n + \lambda^{n+k-1}{}_nC] \frac{\lambda_k p|\gamma|\beta}{(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]} \\
&= p|\gamma|\beta \sum_{k=2}^{\infty} \lambda_k \\
&= p|\gamma|\beta(1 = \lambda_1) \\
&\leq p|\gamma|\beta.
\end{aligned}$$

From Theorem 2.1 $f(z) \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$. Setting $\lambda_k = \frac{(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]}{p|\gamma|\beta} a^k$

and $\lambda_1 = 1 - \sum\limits_{k=2}^{\infty} \lambda_k$

$$\begin{aligned}
\sum_{k=1}^{\infty} \lambda_k f_k(z) &= \lambda_1 f_1(z) + \sum k = 2\lambda_k f_k(z) \\
&= \left(1 - \sum_{k=2}^{\infty} \lambda_k\right) z + \sum_{k=2}^{\infty} \lambda_k \left(z + \frac{p|\gamma|\beta}{(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]} z^k\right) \\
&= 1 - \sum_{k=2}^{\infty} \lambda_k \frac{p|\gamma|\beta}{(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]} z^k \\
&= 1 + \sum_{k=2}^{\infty} \frac{(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_k}{p|\gamma|\beta} \\
&\quad \frac{p|\gamma|\beta}{(p+(k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]} z^k \\
&= z + \sum_{k=2}^{\infty} a_k z^k \\
&= f(z).
\end{aligned}$$

Hence $f(z) = \sum\limits_{k=1}^{\infty} \lambda_k f_k(z)$.

5. Integral Mean Inequality for Differential Operator

Theorem 5.1 : $f(z) \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$ and suppose that

$$\sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]|a_k| \leq \frac{p|\gamma|\beta}{[p+\mu(j-1)]}. \tag{5.1}$$

Also let the function

$$f_j(z) = z + \frac{p|\gamma|\beta}{(p+(j-1)\mu)(1-\lambda)j^n + \lambda^{n+j-1}_{\ \ n}C]} z^j \quad (j \geq 2). \tag{5.2}$$

If there exists an analytic function $w(z)$ given by

$$w(z)^{j-1} = \frac{(p+(j-1)\mu)}{p|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda^{n+k--1}_{\ \ \ n}C]a_k z^{k-1}. \tag{5.3}$$

Then for $z = re^{i\theta}$ with $0 < r < 1$

$$\int_0^{2\pi} |D_\lambda^n f(z)|^t d\theta \leq \int_0^{2\pi} |D_\lambda^n f_j(z)|^t d\theta \quad (0 \leq \lambda \leq 1, t > 0)$$

where D_λ^n is differential operator defined in (1.6).

Proof : We have from definition (1.6)

$$D_\lambda^n f(z) = 1 + \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_k z^k$$

$$D_\lambda^n f_j(z) = z + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} z^j.$$

For $z = re^{i\theta}$ with $0 < r < 1$ we have to show that

$$\int_0^{2\pi} \left| 1 + \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_z^{k-1} \right|^t d\theta$$
$$\leq \int_0^{2\pi} \left| 1 + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} z^{j-1} \right|^t d\theta \quad (t > 0).$$

By applying Littlewoods subordination theorem, it would sufficient to show that

$$1 + \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_k z^{k-1} \prec 1 + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} z^{j-1}.$$

That is $t(z) \prec h(z)$ where

$$t(z) = 1 + \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_k z^{k-1}$$

$$h(z) = 1 + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} z^{j-1}.$$

That is we want to show that $t(z) = h(w(z))$, $w(0) = 0$ and $|w(z)| \leq 1$

$$\begin{aligned}
h(w(z)) &= 1 + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} w(z)^{j-1} \\
&= 1 + \frac{p|\gamma|\beta}{(p+(j-1)\mu)} \frac{(p(j-1)\mu)}{p|\gamma|\beta} \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_k z^{k-1} \\
&= 1 + \sum_{k=2}^{\infty}[(1-\lambda)k^n + \lambda^{n+k-1}_{\ \ n}C]a_k z^{k-1} \\
&= t(z).
\end{aligned}$$

Therefore $h(w(z)) = t(z)$ and $w(0) = 0$.

Moreover we prove that analytic function $|w(z)| < 1, z \in U$

$$\begin{aligned} |w(z)^{j-1}| &= \left| \frac{(p+(j-1)\mu)}{p|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda^{n+k-1}{}_{n}]a_k z^{k-1} \right| \\ &\leq \frac{(p+(j-1)\mu)}{p|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda^{n+k-1}{}_{n}C]|a_k||z^{k-1}| \\ &\leq |z| \frac{(p(j-1)\mu)}{p|\gamma|\beta} \sum_{k=2}^{\infty} [(1-\lambda)k^n + \lambda^{n+k-1}{}_{n}C]|a_k| \\ &\leq |z| < 1 \quad \text{by hypothesis(5.1).} \end{aligned}$$

Hence proved.

6. Convolution Theorems

Definition 6.1 : If $f(z) = z + g(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, then hadmad product is defined as given bellow

$$f * g = z + \sum_{k=2}^{\infty} (a_k b_k) z^k. \tag{6.1}$$

Theorem 6.1 : Let $f, g \in \tilde{K}(D_{\lambda}^{m}; \gamma, \mu, m, \beta)$ where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, with $a_k \geq 0$, $b_k \geq 0$ and $(a_k b_k)^{\frac{1}{2}} < 1$. Then $f * g \in \tilde{K}_p(D_{\lambda}^{m}; \gamma, \mu, m, \beta)$.

Proof : We have $f \in \tilde{K}(D_{\lambda}^{m}; \gamma, \mu, m, \beta)$ if and only if

$$\sum_{k=2}^{\infty} [p + \mu(k-1)][(1-\lambda)k^n + \lambda^{n+k-1}{}_{n}C]|a_k| \leq p|\gamma|\beta$$

$g \in \tilde{K}(D_{\lambda}^{m}; \gamma, \mu, m, \beta)$ if and only if

$$\sum_{k=2}^{\infty} [p + \mu(k-1)][(1-\lambda)k^n + \lambda^{n+k-1}{}_{n}C]|b_k| \leq p|\gamma|\beta$$

By Cauchy Schwarz inequality

$$\sum_{k=2}^{\infty} (t_i k |a_k| t_k |b_k|)^{\frac{1}{2}} \leq \left(\sum_{k=2}^{\infty} t_k |a_k| \right)^{\frac{1}{2}} \left(\sum_{k=2}^{\infty} (t_k |b_k|) \right)^{\frac{1}{2}}$$

where $t_k = (p + (k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_{n}C]$

$$\sum_{k=2}^{\infty} (p + (k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_{n}C]|a_k b_k|^{\frac{1}{2}} \leq p|\gamma|\beta. \tag{6.2}$$

By assumption $(a_k b_k)^{\frac{1}{2}} < 1$. Then

$$a_k b_k < (a_k b_k)^{\frac{1}{2}}. \tag{6.3}$$

Thus from (6.2) and (6.3)

$$\sum_{k=2}^{\infty} (p+k-1)\mu)[(1-\lambda)k^n + \lambda^{n+k-1}{}_nC]|a_k b_k| < p|\gamma|\beta.$$

Hence $f * g \in \tilde{K}_p(D_\lambda^m; \gamma, \mu, m, \beta)$.

References

[1] Al-Oboudi F. M., On univalent functions defined by a generalized Săală̆gean operator, Internat. J. Math. and Mathematical Sciences, 27 (2004), 1429-1436.

[2] Duren P. L., Univalent Functions, Grundlehren der Mathematischen Wissenchaften, 259, Springer-Verlag, New York, USA, (1983).

[3] Eker S. S. and G*ü*ney H. *Ö*, A new subclass of analytic functions involving Al-Oboudi differential operator, Journal of Inequalities and Applications, (2008), Art. ID. 452057.

[4] Khairnar S. M. and More M., A class of analytic functions defined by Hurwitz-Lerch Zeta function, Internat. J. Math. and Computation, 1(8) (2008), 106-123.

[5] Khairnar S. M. and More M., A new class of analytic and multivalent functions defined by Subordination property with applications to generalized hyper geometric functions, Gen. Math., (to appear in 2009).

[6] Kihc O. O., Sufficient conditions for subordination of multivalent functions, Journal of Inequalities and Applications, (2008), Art. ID. 374756.

[7] Littlewood J. E., On inequalities in the theory of functions, Proc. London Math. Soc., 23 (1925), 481-519.

[8] Milleer S. S. and Mocanu P. T., Differential Subordinations, Theory and Applications, 225, of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, (2000).

[9] Murugusundaramoorthy G. Rosy T. and Muthunagai K., Carlson-Shaffer operator and their applications to certain subclass of uniformly convex functions, Gen. Math., 15(4) (2007), 131-143.

[10] Raina R. K. and Bansal D., Some properties of a new class of analytic functions defined in terms of a Hadamard product, J. Inequal. Pure and Appl. Math., 9(1) (2008), Art. 22. [ONLINE: http: ——jipam.vu.edu.au— article.php?sid=957]

[11] Sălăgean G. S., Subclass of Univalent Functions in Complex Analysis, 5th Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Vol. 1013 of Lecture Notes in Mathematics, Springer, Berlin, Germany, (1983), 362-372.

[12] Silverman H., Partial sums of star like and convex functions, J. Math. Anal. and Appl., 209 (1997), 221-227.

[13] Silvia E. M., Partial sums of convex functions of order _, Houston J. Math., Math. Soc., 11(3) (1985), 397-404.

[14] Khairnar S. M. and More Meena, On certain subclass of analytic functions involving Al-Oboudi differential operator, 2 (2008), 1739-1748.