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BEST PROXIMITY POINTS : EXISTENCE AND CONVERGENCE THEOREMS FOR p-CYCLIC WEAK ϕ CONTRACTIONS MAPPINGS

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Abstract

We introduce a new class of mappings called $p-cyclic\ weak\ \phi-contractions$ which contains the $p-cyclic\ \phi-contractions$ mappings as a subclass. Then, convergence and existence of best proximity points for $p-cyclic\ weak\ \phi-contractions$ mappings are obtained. Moreover, results are generalizations of the results of calogero vetro (2010)[30].

1. Introduction

In 1922, Banach[3] stated that every contraction on a complete metric space has a unique fixed point. This theorem is known as Banach contraction mapping principle or Banach fixed point theorem. Banach's theorem preserves its importance in fixed point theory and has applications not only in many branches of mathematics but also in

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in economics. In particular, in micro economics, for the Nash equilibria, fixed point theorems are used (see e.g [20,4]).

In 1969, Boyd and wong[5] gave the definition of ϕ-contraction: A self mapping T on a metric space X is called ϕ-contraction if there exists an upper semi-continuous function $\phi : [0, +\infty) \to [0, +\infty)$ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) \text{ for all } x, y \in X.$$

Later, in 1997, Alber and Guerre-Delabriere [1], introduced the definition of weak ϕ-contraction: A self mapping T on a metric space X is called weak ϕ-contraction if for each $x, y \in X$, there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ such that

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$$

In addition, Alber and Guerre-Delabriere defined weak ϕ-contraction on Hilbert spaces and proved the existence of fixed points in Hilbert spaces. Rhoades[25] showed that most of the results in [1] are also valid for arbitrary Banach spaces.

Notice that if ϕ is a lower semi-continuous mapping then $\phi(u)$=u-$\phi(u)$ becomes ϕ-contraction[5]. The notions ϕ-contraction and weak ϕ-contraction have been studied by many authors,(e.g; [12, 26, 27, 28, 14, 15, 16].)

Cyclic maps were defined by Kirk-Srinivasan-Veeramani in 2003. They stated the following Theorem (see[17], Theorem 1.1).

Theorem 1.1 : Let A and B be non-empty closed subsets of a complete metric space (X,d). Suppose that T: A $\cup$ B $\to$ A $\cup$ B is a map satisfying T(A)$\subset$B and T(B)$\subset$ A and there exists k$\in$(0,1) such that d(Tx,Ty)$\leq$kd(x,y) for all x$\in$A and y$\in$B. Then, T has a unique fixed point in A$\cap$B.

(see[2,7]). Let A and B be non-empty closed subsets of a metric space (X,d) and ϕ: $[0, +\infty) \to [0, +\infty)$ be a strictly increasing map. A map T:A$\cup$B$\to$A$\cup$B is a called a cyclic weak ϕ-contraction if T(A)$\subset$B and T(B)$\subset$A and

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) + \phi(d(A, B))$$

for all $x \in A$ and $y \in B$ where d(A,B)=inf{d(a,b):a$\in$ A,b$\in$B}.

A point x$\in$A$\cup$B is called a best proximity point if d(x,Tx)=d(A,B). Further, if $\alpha \in$(0,1) and ϕ(t)=(1-α)t, then T is called cyclic contraction (see[10]). Rezapour-Derafshpour-shahzad (see[23], also[24]) stated the following theorem:

Theorem 1.2 : Let (X,d,$\leq$) be an ordered metric space, A and B be non-empty subsets of X and T:A∪B→A∪B be decreasing, cyclic weak ϕ-contraction, that is, T satisfies (Cyclic weak ϕ contraction). Suppose there exists x_0 ∈A such that $x_0 \leq T^2x_0 \leq Tx_0$.Define x_{n+1} =Tx_n and d_n:=d(x_{n+1},x_n) for all n∈ N. Then d_n →d(A,B).

(see[7]). Let A and B be non-empty closed subsets of a metric space (X,d) and ϕ: $[0,+\infty) \to [0,+\infty)$ be a strictly increasing map. A map T:A∪B→A∪B is a called a Kannan type cyclic weak ϕ-contraction if T(A)⊂B and T(B)⊂A and

$$d(Tx,Ty) \leq u(x,y) - \phi(u(x,y)) + \phi(d(A,B))$$

for all $x \in A$ and $y \in B$ where u(x,y)=$\frac{1}{2}$[d(x,Tx)+d(y,Ty)] and d(A,B)=inf{d(a,b):a∈ A,b∈B}.

2. Preliminaries

In this section we give some basic definitions and concepts which are useful and related to the context of our results.

Definition 2.1 : Let $A_1, A_2, \cdots, A_p$ be nonvoid subsets of a metric space (X,d). A mappings $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ is said to be a p-cyclic mapping if $T(A_i) \subset A_{i+1}$ for $i = 1, 2, \cdots, p$ where $A_{p+1} = A_1$. A point $x \in A_i$ is said to be a Best proximity point if $d(x,Tx) = d(A_i, A_{i+1})$.

Lemma 2.1 (3.3 [29]) : Let (X,d) be a metric space and let $A_1, A_2, \cdots, A_p$ be non-void subsets of X. if $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ is a p-cyclic non-expansive mapping then $d(A_i, A_{i+1}) = d(A_{i+1}, A_{i+2}) = d(A_1, A_2)$for $i = 1, 2, ..., p$.

Definition 2.2 [11] : Let $A_1, A_2, \cdots, A_p$ be nonvoid subsets of a metric space (X,d).A p-cyclic mapping T on $\cup_{i=1}^p A_i$ is called a contraction mapping if there exists $k \in (0,1)$ such that $d(Tx,Ty) \leq kd(x,y) + (1-k)d(A_i, A_{i+1})$
$\forall x \in A_i, y \in A_{i+1}, i = 1, 2, ..., p$.

Definition 2.3 : Let (X,d) be a metric space and let $A_1, A_2, \cdots, A_p$ be nonvoid subsets of X. A p-cyclic mapping T on $\cup_{i=1}^p A_i$ is called a weak ϕ-contraction if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ such that

$$d(Tx,Ty) \leq v(x,y) - \phi(v(x,y)) + \phi(d(A_i, A_{i+1}))$$

for all $x \in A_i$ and $y \in A_{i+1}$ where v(x,y)=$\frac{3}{4}$[d(Tx,Ty)+d(x,y)]and
$d(A_i, A_{i+1})$=inf{d(a,b):a∈ A_i,b∈B_{i+1}}

Lemma 2.2 : Let (X,d) be a metric space and let $A_1, A_2, \cdots, A_p$ be nonvoid subsets of X. If $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a p-cyclic ϕ-contraction then

(i) $-\phi(d(x,y)) + \phi(d(A_i, A_{i+1})) \leq 0$ for all $x \in A_i$ and $y \in A_{i+1}$, i=1,2,....,p

(ii)$d(Tx, Ty) \leq d(x,y)$ for all $x \in A_i$ and $y \in A_{i+1}$, i=1,2,....,p for all $x \in A_i$ and $y \in A_{i+1}$.

Theorem 2.3 : Let $A_1, A_2, \cdots, A_p$ be nonvoid subsets of metric space X.Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a p-cyclic ϕ-contraction. For $x_0 \in \cup_{i=1}^{p} A_i$ Then $d(T^{pn}x, T^{pn+1}y) \to d(A_i, A_{i+1})$ as $n \to \infty$ for all $x, y \in A_i$.

Theorem 2.4 : Let $A_1, A_2, \cdots, A_p$ be nonvoid subsets of metric space X.Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a p-cyclic ϕ-contraction.If for $x \in A_i$,the sequence $\{T^{pn}x\}$ has a convergent subsequence in $\{T^{pn_k}x_0\}$ convergent to a point $x \in A_i$ then $d(x, Tx) = d(A_i, A_{i+1})$.

Theorem 2.5 : let $A_1, A_2, \cdots, A_p$ be nonvoid subsets of a uniformly convex Banach space X such that A_i, A_{i+1} is convex and let Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be p-cyclic ϕ-contraction.Given $x_0 \in A_i$ then for every $\epsilon > 0$ there exists n_ϵ such that $\| T^{pm}x_0 - T^{pn+1}x_0 \| < d(A_i, A_{i+1}) + \epsilon$ for all $m > n \geq n_\epsilon$.

Theorem 2.6 : let $A_1, A_2, \cdots, A_p$ be nonvoid subsets of a uniformly convex Banach space X such that A_i, A_{i+1} is convex and let Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be p-cyclic ϕ-contraction.If A_i is closed and $d(A_i, A_{i+1}) = 0$.Then T has a unique fixed point $x \in \cap_{i=1}^{p}$ and $T^{pn}x_0 \to x (T^n x_0 \to x)$ as $n \to \infty$ for all $x_0 \in Ai$.

3. Results for p-cyclic weak ϕ Contraction

Now let us state our main result.

Example 1 : Let $X = R$ with the usual metric and let $A_1, A_2, \cdots, A_p = [0, 1]$. The mapping $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ defined by $Tx = \frac{x}{1+x}$ is a p-cyclic weak ϕ contraction if we choose a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) = \frac{t^2}{(1+t)}$ T is not a p-cyclic ϕ-contraction.

Example 2 : Let $X = R$ with usual metric and let $A_1 = [0, 1], A_2 = [-1, 0], A_3 = [1, 2]$. The mapping $T : \cup_{i=1}^{3} A_i \to \cup_{i=1}^{3} A_i$ defined by $Tx = \frac{-x}{4(1+|x|)}$ is a 3-cyclic weak ϕ-contraction if we choose $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) = \frac{t^2}{(1+t)}$.

Lemma 3.1 : Let (X,d) be a metric space and let $A_1, A_2, \cdots, A_p$ be nonvoid subsets of X. If $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a p-cyclic weak ϕ-contraction then

(i) $-\phi(v(x,y)) + \phi(d(A_i, A_{i+1})) \leq 0$ for all $x \in A_i$ and $y \in A_{i+1}$, i=1,2,$\cdots$,p

(ii) $d(Tx, Ty) \leq v(x,y)$ for all $x \in A_i$ and $y \in A_{i+1}$, i=1,2,$\cdots$,p.

Proof : Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a p-cyclic weak ϕ-contraction mapping
To prove that
(i) $-\phi(v(x,y)) + \phi(d(A_i, A_{i+1})) \leq 0$ for all $x \in A_i$ and $y \in A_{i+1}$, i=1,2,...,p
clearly $d(A_i, A_{i+1}) \leq v(x,y)$ for all $x \in A_i, y \in A_{i+1}$ since ϕ is a strictly increasing function, $-\phi(v(x,y)) + \phi(v(x,y)) = 0$. Obiviously, $-\phi(v(x,y)) + \phi(d(A_i, A_{i+1})) < 0$
hence,$-\phi(v(x,y)) + \phi(d(A_i, A_{i+1})) \leq 0$ for all $x \in A_i$ and $y \in A_{i+1}$, i=1,2,....,p
(ii)to prove that $d(Tx, Ty) \leq v(x,y)$ for all $x \in A_i$ and $y \in A_{i+1}$, i=1,2,....,p
from(i) we get

$$-\phi(v(x,y)) + \phi(d(A_i, A_{i+1})) \leq 0.$$

From the definition of (2.3)

$$d(Tx, Ty) \leq v(x,y) - \phi(v(x,y)) + \phi(d(A_i, A_{i+1}))$$

for all $x \in A_i$ and $y \in A_{i+1}$ where v(x,y)=$\frac{3}{4}$[d(Tx,Ty)+d(x,y)] and $d(A_i, A_{i+1}) = inf\{d(a,b) : a \in A_i, b \in A_{i+1}\}$
Therefore we get $d(Tx, Ty) \leq v(x,y)$ for all $x \in A_i$ and $y \in A_{i+1}$, i=1,2,....,p, □

Theorem 3.1 : Let $A_1, A_2, \cdots, A_p$ be nonvoid subsets of metric space X.Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a p-cyclic weak ϕ-contraction. For $x_0 \in \cup_{i=1}^{p} A_i$.
Then $d(Tx_{pn}, Ty_{pn+1}) \to d(A_i, A_{i+1})$ as $n \to \infty$ for all $x, y \in A_i$.

Proof : Let $d_n = d(Tx_{pn}, Tx_{pn+1})$ for all $n \geq 1$. It follows from Lemma 3.1(ii) that $\{d_n\}$ is decreasing and bounded.
Thus $\lim_{n \to \infty} d_n = t_0$ for some $t_0 \geq d(A_i, A_{i+1})$. If $t_0 = d(A_i, A_{i+1})$ there is nothing to prove. So, we assume that $t_0 > d(A_i, A_{i+1})$
Now, we have

$$\begin{aligned} d_{n+1} &= d(Tx_{p(n+1)}, Ty_{p(n+1)+1}) \leq \cdots \leq d(Tx_{pn+1}, Ty_{pn+2}) \\ &\leq d_n - \phi(d_n) + \phi(d(A_i, A_{i+1})) \end{aligned}$$

and hence

$$\phi(d(A_i, A_{i+1})) \leq \phi(t_0) \leq \phi(d_n) \leq d_n - d_{n+1} + \phi(d_0) \quad (1)$$

for all $n \geq 1$ since ϕ is a strictly increasing function $d_n \geq t_0 \geq d(A_i, A_{i+1})$. for all $n \geq 1$. It follows from (1) that

$$\lim_{n\to\infty} \phi(d_n) = \phi(t_0) = \phi(d(A_i, A_{i+1}))$$

As, ϕ is strictly increasing function we have $t_0 = d(A_i, A_{i+1})$. □

Corollary 3.3 : Let $A_1, A_2, ..., A_p$ be nonvoid subsets of metric space X.Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a p-cyclic weak ϕ-contraction. Then

(i) $d(Tx_n, Tx_{n+1}) \to d(A_1, A_2)$ as $n \to \infty$ for all $x_0 \in A_i$, i=1,2,...,p

(ii) $d(Tx_{p(n+1)}, Tx_{pn+1}) \to d(A_i, A_{i+1})$ as $n \to \infty$for all $x_0 \in A_i$, i=1,2,...,p.

Proof : (i)By lemma 2.1 $d(A_1, A_2) = d(A_i, A_{i+1})$ for all i=1,2,...,p
since T is a weak ϕ - contraction.
By Lemma 3.1 the sequence $\{d(Tx_n, Tx_{n+1})\}$ is decreasing and so (i) follows from Theorem 3.2.
(ii) By Theorem 3.2 if $T = Tx_p$ and we obtain (ii) that is,
$d(Tx_{p(n+1)}, Tx_{pn+1}) \to d(A_i, A_{i+1})$ as $n \to \infty$for all $x_0 \in A_i$, i=1,2,....,p. □

Theorem 3.4 : Let $A_1, A_2, ..., A_p$ be nonvoid subsets of metric space X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a p-cyclic weak ϕ-contraction. If for $x \in A_i$, the sequence $\{TX_{pn}\}$ has a convergent subsequence in $\{Tx_{pn_k}\}$ convergent to a point $x \in A_i$ then $d(x, Tx) = d(A_i, A_{i+1})$.

Proof : From

$$\begin{aligned} d(A_i, A_{i+1}) &= d(A_{i-1}, A_i) \leq d(x, Tx_{pn_k-1}) \\ &\leq d(x, Tx_{pn_k}) + d(Tx_{pn_k-1}, Tx_{pn_k}) \end{aligned}$$

for all $k \geq 1$, it follows by Corollary 3.3(i).
That $d(x, Tx_{pn_k-1}) \to d(A_i, A_{i+1})$.
Since $d(A_i, A_{i+1}) \leq d(Tx_{pn_k}, Tx) \leq d(Tx_{pn_k-1}, x)$ for all $k \geq 1$, as $k \to \infty$ it follows that hence $d(x, Tx) = d(A_i, A_{i+1})$. □

Lemma 3.5 : Let $A_1, A_2, ..., A_p$ be nonvoid subsets of a uniformly convex Banach space X such that A_i is convex and let Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be p-cyclic weak ϕ-contraction then $\| Tx_{p(n+1)} - Tx_{pn} \| \to 0$ and
$\| Tx_{p(n+1)+1} - Tx_{pn+1} \| \to 0$ as $n \to \infty$ for all $x \in A_i$.

Proof : To show that $\| Tx_{p(n+1)} - Tx_{pn} \| \to 0$, $n \to \infty$
we assume the contrary, then there exists $\epsilon_0 > 0$ such that for all $k \geq 1$ then there exists $n_k \geq k$ so that

$$\| Tx_{p(n_k+1)} - Tx_{pn_k} \| \geq \epsilon_0.$$

Choose $\alpha \in (0, 1)$ such that $\frac{\epsilon_0}{\alpha} > d(A_i, A_{i+1}) = d_0$ and choose ϵ such that

$$0 < \epsilon < min\{\frac{\epsilon_0}{\alpha} - d_0, \frac{\delta(\alpha) d_0}{1 - \delta(\alpha)}\}.$$

Now by Corollary 3.3 (ii) and Theorem 3.2, there exists n_ϵ such that

$$\| Tx_{p(n_k+1)} - Tx_{pn_k+1} \| < d_0 + \epsilon$$

and

$$\| Tx_{pn_k} - Tx_{pn_k+1} \| < d_0 + \epsilon$$

for all $n_k \geq n_\epsilon$. It follows from the uniform convexity of X that

$$\| Tx_{pn_k} + Tx_{p(n_k+1)} - Tx_{pn_k+1} \| \leq (1 - \delta(\frac{\epsilon_0}{d_0 + \epsilon}))(d_0 + \epsilon)$$

for all $n_k \geq n_\epsilon$.As $\frac{(Tx_{pn_k} + Tx_{p(n_k+1)})}{2} \in A_i$
the choice of ϵ and the fact that δ is strictly increasing imply that
$\| \frac{Tx_{pn_k} + Tx_{p(n_k+1)}}{2} - Tx_{pn_k+1} \| < d_0$ for all $n_k \geq n_\epsilon$ which is a contradiction.
A similar argument shows $\| Tx_{p(n+1)+1} - Tx_{pn+1} \| \to 0$ as $n \to \infty$. □

Theorem 3.6 : Let $A_1, A_2, ..., A_p$ be nonvoid subsets of a uniformly convex Banach space X such that A_i, A_{i+1} is convex and let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be p-cyclic weak ϕ-contraction. Given $x_0 \in A_i$ then for every $\epsilon > 0$ there exists n_ϵ such that $\| Tx_{pm} - Tx_{pn+1} \| < d(A_i, A_{i+1}) + \epsilon$ for all $m > n \geq n_\epsilon$.

Proof : Suppose the contrary .Then there exists $\epsilon_0 > 0$ such that for each $k \leq 1$ there exists $m > n \geq k$ satisfying

$$\| Tx_{pm_k} - Tx_{pn_k+1} \| \geq d(A_i, A_{i+1}) + \epsilon_0 \quad \text{and} \tag{2}$$

$$\| Tx_{p(m_k-1)} - Tx_{pn_k+1} \| \leq d(A_i, A_{i+1}) + \epsilon_0 \tag{3}$$

It follows from (2)the triangle inequality and (3) that

$$\begin{aligned}
&d(A_i, A_{i+1}) + \epsilon_0 \leq \| Tx_{pm_k} - Tx_{pn_k+1} \| \\
&\leq \| Tx_{pm_k} - Tx_{p(m_k-1)} \| + \| Tx_{p(m_k-1)} - Tx_{pn_k+1} \| \\
&< \| Tx_{pm_k} - Tx_{p(m_k-1)} \| + d(A_i, A_{i+1}) + \epsilon_0
\end{aligned}$$

Letting $k \to \infty$ Lemma 3.5 implies

$$\lim_{n\to\infty} \| Tx_{pm_k} - Tx_{pn_k+1} \| = d(A_i, A_{i+1}) + \epsilon_0 \tag{4}$$

Applying the triangle inequality Lemma 3.1 (i,ii) and the cyclic ϕ-contraction property of T, we obtain

$$\begin{aligned}
\| Tx_{pm_k} - Tx_{pn_k+1} \| &\leq \| Tx_{pm_k} - Tx_{pm_k+p} \| + \| Tx_{p(m_k+1)} - Tx_{p(n_k+1)+1} \| \\
&\quad + \| Tx_{p(n_k+1)+1} - Tx_{pn_k+1} \| \\
&\leq \| Tx_{pm_k} - Tx_{p(m_k+1)} \| + \| Tx_{pm_k+1} - Tx_{pn_k+2} \| \\
&\quad + \| Tx_{p(n_k+1)+1} - Tx_{pn_k+1} \| \\
&\leq \| Tx_{pm_k} - Tx_{p(m_k+1)} \| + \| Tx_{pm_k} - Tx_{pn_k+1} \| \\
&\quad - \phi(\| Tx_{pm_k} - Tx_{pn_k+1} \|) + \phi(d(A_i, A_{i+1})) \| \\
&\quad + \| Tx_{p(n_k+1)+1} - Tx_{pn_k+1} \| \\
&\leq \| Tx_{pm_k} - Tx_{p(m_k+1)} \| + \| Tx_{pm_k} - Tx_{pn_k+1} \| \\
&\quad + \| Tx_{p(n_k+1)+1} - Tx_{pn_k+1} \| .
\end{aligned}$$

Letting $k \to \infty$ and using (4)and Lemma 3.5 we obtain

$$\begin{aligned}
d(A_i, A_{i+1}) + \epsilon_0 &\leq d(A_i, A_{i+1}) + \epsilon_0 - \lim_{k\to\infty} \phi(\| Tx_{pm_k} - Tx_{pn_k+1} \|) \\
&\quad + \phi(d(A_i, A_{i+1}) \\
&\leq d(A_i, A_{i+1}) + \epsilon_0
\end{aligned}$$

and hence,

$$\lim_{k\to\infty} \phi(\| Tx_{pm_k} - Tx_{pn_k+1} \|) = \phi(d(A_i, A_{i+1})) \tag{5}$$

Since ϕ is strictly increasing, it follows from (1) and (5) that

$$\begin{aligned}\phi(d(A_i, A_{i+1}) + \epsilon_0) &\leq \lim_{k\to\infty} \phi(\| Tx_{pm_k} - Tx_{pn_k+1} \|) \\ &= \phi(d(A_i, A_{i+1})) < d(A_i, A_{i+1}) + \epsilon_0\end{aligned}$$

which is a contradiction. □

Theorem 3.7 : Let $A_1, A_2, ..., A_p$ be nonvoid subsets of a uniformly convex Banach space X such that A_i, A_{i+1} is convex and let Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be p-cyclic weak ϕ-contraction.If A_i is closed and $d(A_i, A_{i+1}) = 0$.Then T has a unique fixed point $x \in \cap_{i=1}^{p}$ and $Tx_{pn} \to x(Tx_n \to x)$ as $n \to \infty$ for all $x_0 \in Ai$.

Proof : Choose ϵ by the Theorem 3.2 and 3.4 there exists N_1 such that $\| Tx_{pn} - Tx_{pn+1} \| \leq \epsilon$ for all $n \geq N_1$.

By Theorem 3.6, there exists N_2 such that

$$\| Tx_{pm} - Tx_{pn+1} \| \leq \epsilon$$

for every $m > n \geq N_2$. Let $N = max\{N_1, N_2\}$ it follows that

$$\| Tx_{pm} - Tx_{pn} \| \leq \| Tx_{pm} - Tx_{pn+1} \| + \| Tx_{pn+1} - Tx_{pn} \| < \epsilon$$

for all $m > n \geq N_2$.

Thus $\{Tx_{pn}\}$ is a Cauchy sequence in A_i. Now if A_i is closed.

$Tx_{pn} \to x \in A_i$ as $n \to \infty$.

By theorem $\| x - Tx \| = 0$ and x is a fixed point of T and hence $x \in \cap_{i=1}^{p} A_i$.

To show that the uniqueness of x, we assume that y is another fixed point of T since

$$\| x - y \| = \| Tx - Ty \| \leq \| x - y \| - \phi(\| x - y \|) + \phi(0)$$

it follows that $\phi(\| x - y \|) = \phi(0)$ and so x=y.

The sequence $\{\| x - Tx_n \|\}$ is decreasing and bounded, then we conclude that $Tx_{pn} \to x$ as $n \to \infty$. □

References

[1] Alber Ya. I., Guerre-Delabriere S., Principle of weakly contractive maps in Hilbert space In: New Results in Operator Theory, Advances and Appl. (Eds: I. Gohberg and Yu. Lyubich,) 722, Birkhauser, Basel, Oper. Theory Adv. Appl., 98 (1997).

[2] Al-Thafai M. A., Shahzad N., Convergence and existence for best proximity points, Nonlinear Analysis, 70 (2009), 3665-3671.

[3] Banach S., Sur les operations dans les ensembles abstraits et leur application aux equations itegrales, Fund. Math., 3 (1922), 133-181.

[4] Border K. C., Fixed Point Theorems With Applications to Economics and Game Theory, Cambridge University Press, Newyork, (1985).

[5] Boyd D. W. and Wong S. W., On nonlinear contaractions, Proc. Amer. Math. Soc., 20 (1969), 458-464.

[6] Ciric Lj. B., On contraction type mappings Math. Balkancia, 1 (1971), 52-57.

[7] Erdal Karapinar, Best proximity points of Kannan type cyclic weak ϕ-contractions in ordered metric spaces, Ovidius Constanta, 20(3) (2012), 51-64.

[8] Ciric Lj. B., On some maps with a nonunique fixed point. Publ. Inst. Math., 17 (1974), 52-58.

[9] Ding H. S., Li L. and Radenovic S., Bhaskar-Lakshmikantham-Ciric-type theorems in partially ordered metric space, Preprint.

[10] Eldred A. A., Veeramani P., Proximal pointwise contraction. Topology and its Applications, 156 (2009), 2942-2948.

[11] Eldred A. A., Veeramani P., Existence and convergence of best proximity points, J. Math. Anal. Appl., 323 (2006), 1001-1006.

[12] Hussain N., Jungck G., Common fixed point and invariant approximation results for noncommuting generalized (f,g)-nonexpansive maps, J. Math. Anal. Appl., 321 (2006), 851-861.

[13] Karapinar E., Some nonunique fixed point theorems of ciric type on cone metric spaces, Abst. Appl. Anal., (2010), Article id 123094, 14 pages (2010), doi:10.1155/2010/123094.

[14] Karapinar E., Fixed point theory for cyclic weak ϕ-contraction, Appl. Math. Lett., 24(6) (2011), 822-825.

[15] Karapinar E., Best proximity points of cyclic mappings, Appl. Math., Lett., 335(Issue 1,1) (November 2007), 79-92.

[16] Karapinar E., Erhan I. M. and Ulus A. Y., Fixed point theorem for cyclic maps on partial metric spaces, Appl. Math. Inf. Sci., 6(1) (2012), 239-244.

[17] Kirk W. A., Srinivasan P. S. and Veeramani P., Fixed points for mapping satisfying cyclic contractive conditions, Fixed Point Theory, 4 (2003), 79-89.

[18] Mitrovic Z. D., A coupled best approximations theorem in normed spaces, Nonlinear Anal., 72 (2010), 4049-4052.

[19] Niteo J. J., R. L and Pouse, R. Rodriguez-Lopez, Fixed point theorems in ordered abstract spaces, Proc. Amer. Math. Soc., 137 (2007), 2505-2517.

[20] Ok E. A., Real Analysis with Economic Applications, Princeton University Press, (2007).

[21] Petrusel G., Cyclic representations and periodic points, Studia Univ. Babes-Bolyai Math., 50 (2005), 107-112.

[22] Petrusel G. and Rus I. A., Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc., 134 (2006), 411-418.

[23] Best proximity point of cyclic ϕ-contractions in Ordered metric spaces, Topological Methods in Nonlinear Analysis, 37 (2011), 193-202.

[24] Rezapour Sh., Derafshpour M., Shahzad N., Best proximity point of cyclic ϕ-contractions on reflexive banach spaces, Fixed Point Theory and Applications, (2010), Article id 946178, 7 pages(2010).

[25] Rhoades B. E., Some theorems on weakly contractive maps. Nonlinear Anal., 47(4) (2001), 2683-2693.

[26] Song Y., Coincidence points for noncommuting f-weakly contractive mappings, Int. J. Comput. Appl. Math., 2(1) (2007), 51-57.

[27] Song Y., Xu S., A note on common fixed-points for Banach operator pairs, Int. J. Contemp. Math. Sci., 2 (2007), 1163-1166.

[28] Zhang Q. and Song Y., Fixed point theory for generalized ϕ-weak contractions, Appl. Math. Lett., 22(1) (2009), 75-78.

[29] Karpagam S., Agrawal S., Best proximity point theorems for p-cyclic Meer-keeler contractions.Fixed point theory, Appl. 2009 (2009) Article ID 197308, 9 pages.

[30] Calogero Vetro, Best proximity point : Convergence and existence theorems for p-cyclic mappings, Nonlinear Anal., 73 (2010), 2283-2291.