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SUBCLASSES OF REGULAR AND MULTIVALENT FUNCTIONS DEFINED BY GENERALIZED BERNARDILIBERA-LIVINGSTON INTEGRAL OPERATOR

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Abstract

By means of certain differential operator we introduce and investigate two subclasses $\Re S^q_{n,m}(\lambda, b, \delta)$ and $\mathcal{R}G^q_{n,m}(\lambda, b, \delta)$ of q-valently analytic functions. The various results obtained here for each of these classes. We have attempted to obtain coefficient estimate, distortion theorem, radius of starlikeness, convexity and closure theorem for the classes $\Re S^q_{n,m}(\lambda, b, \delta)$ and $\mathcal{R}G^q_{n,m}(\lambda, b, d)$.

1. Introduction

Let $A(n)$ be the class which contains the functions $h(w)$ normalized by

$$h(w) = w^q - \sum_{x=n+q}^{\infty} a_x w^x, \quad a_x \geq 0 \quad \text{and} \quad n, q \in \mathbb{N} \tag{1.1}$$

which are regular and q-valent in $E = \{w : |w| < 1\}$.

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Key Words and Phrases : *Multivalent function, Coefficient estimate, Distortion theorem, Radius of starlikeness, Differential operator.*

We introduce here Bernardi-Libera-Livingston Integral Operator:

$$\mathcal{F}_q^{\lambda} h(w) = \frac{\lambda + q}{w^{\lambda}} \int_0^w y^{\lambda-1} h(y) dy, \quad (\lambda > -q; w \in E)$$

Simplifying we get

$$\mathcal{F}_q^{\lambda} h(w) = w^q - \sum_{x=n+q}^{\infty} \frac{\lambda + q}{\lambda + x} a_x w^x,$$

$$(\mathcal{F}_q^{\lambda} h(w))^{(m)} = \begin{pmatrix} q \\ m \end{pmatrix} w^{q-m} - \sum_{x=n+q}^{\infty} \begin{pmatrix} x \\ m \end{pmatrix} \left(\frac{\lambda + q}{\lambda + x} \right) a_x w^{x-m}$$

where $\begin{pmatrix} x \\ m \end{pmatrix} = \frac{x(x-1)(x-2)\cdots(x-m+1)}{m!}$.

By using the operator $\mathcal{F}_q^{\lambda} h(w)$, we define new subclass that $h(w)$ is in the subclass $\Re S_{n,m}^q(\lambda, b, \delta)$ if $h(w)$ satisfying the relation

$$\left| \frac{1}{b} \left(\frac{\delta w (\mathcal{F}_q^{\lambda} h(w))^{(m+1)} + \lambda w^2 (\mathcal{F}_q^{\lambda} h(w))^{(m+2)}}{\lambda w (\mathcal{F}_q^{\lambda} h(w))^{(m+1)} + (\delta - \lambda)(\mathcal{F}_q^{\lambda} h(w))^{(m)}} - (q - m) \right) \right| < 1 \tag{1.2}$$

$q \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}, w \in E, q > \max(m, -\lambda), b \in \mathbb{C} \cup \{0\}, \lambda \geq 0, 0 < \delta \leq 1.$

Furthermore a function $h(w)$ is in the class $\Re G_{n,m}^q(\lambda, b, \delta)$ if $wh'(w) \in \Re S_{n,m}^q(\lambda, b, \delta)$.

The objective of the paper is to study some important properties of subclasses $\Re S_{n,m}^q(\lambda, b, \delta)$ and $\Re G_{n,m}^q(\lambda, b, \delta)$. We have investigated coefficient bounds and established distortion theorem, radius of starlikeness, convexity and closure theorem.

2. Coefficient Estimates

Theorem 1 : $h(w) \in A(n)$ and given by (1.1) is in $\Re S_{n,m}^q(\lambda, b, \delta)$ if and only if

$$\sum_{x=n+q}^{\infty} Q(x) a_x \leq |b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q - m - 1) + \delta] \tag{2.1}$$

where

$$Q(x) = \left(\frac{\lambda + q}{\lambda + x} \right) \begin{pmatrix} x \\ m \end{pmatrix} [\lambda(x - m - 1) + \delta)][x - q + |\beta|].$$

Proof : Let $h(w) \in \Re S_{n,m}^q(\lambda, b, \delta)$. Therefore we have

$$\left| \frac{1}{b} \left(\frac{\delta w (\mathcal{F}_q^{\lambda} h(w))^{(m+1)} + \lambda w^2 (\mathcal{F}_q^{\lambda} h(w))^{(m+2)}}{\lambda w (\mathcal{F}_q^{\lambda} h(w))^{(m+1)} + (\delta - \lambda)(\mathcal{F}_q^{\lambda} h(w))^{(m)}} - (q - m) \right) \right| < 1 \tag{2.2}$$

This implies that

$$\left|\frac{1}{b}\left(\frac{\sum_{x=n+q}^{\infty}\binom{x}{m}\left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta][-x+m+q-m]a_x w^{x-m}}{\binom{q}{m}[\lambda(q-m-1)+\delta]w^{q-m}-\sum_{x=n+q}^{\infty}\binom{x}{m}\left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta]a_x w^{x-m}}\right)\right| < 1 \tag{2.3}$$

We know that $Re(w) < |w|$, by letting $w \to 1_-$ through real axis, (2.3) gives

$$\frac{1}{|b|}\left[\frac{\sum_{x=n+q}^{\infty}\binom{x}{m}\left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta][x-q]a_x}{\binom{q}{m}[\lambda(q-m-1)+\delta]w^{q-m}-\sum_{x=n+q}^{\infty}\binom{x}{m}\left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta]a_x x^{x-m}}\right] < 1 \tag{2.4}$$

On simplifying we get

$$\sum_{x=n+q}^{\infty} Q(x)a_x \leq |b|\binom{q}{m}[\lambda(q-m-1)+\delta]$$

where

$$Q(x) = \left(\frac{\lambda+q}{\lambda+x}\right)\binom{x}{m}[\lambda(x-m-1)+\delta)][x-q+|b|].$$

Hence we get (2.1).

Now we will prove the converse, by (1.2) and putting $|w| = 1$ we see that

$$\left|\left(\frac{\delta w(\mathcal{F}_q^{\lambda}h(w))^{(m+1)}+\lambda w^2(\mathcal{F}_q^{\lambda}h(w))^{(m+2)}}{\lambda w(\mathcal{F}_q^{\lambda}h(w))^{(m+1)}+(\delta-\lambda)(\mathcal{F}_q^{\lambda}h(w))^{(m)}}-(q-m)\right)\right|$$

$$=\left|\left(\frac{\sum_{x=n+q}^{\infty}\binom{x}{m}\left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta][-x+m+q-m]a_x w^{x-m}}{\binom{q}{m}[\lambda(q-m-1)+\delta]w^{q-m}-\sum_{x=n+q}^{\infty}\binom{x}{m}\left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta]a_x w^{x-m}}\right)\right|$$

$$\leq |b|\frac{\binom{q}{m}[\lambda(q-m-1)+\delta]w^{q-m}-\sum_{x=n+q}^{\infty}\binom{x}{m}\left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta]a_x w^{x-m}}{\binom{q}{m}[\lambda(q-m-1)+\delta]w^{q-m}-\sum_{x=n+q}^{\infty}\binom{x}{m}\left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta]a_x w^{x-m}} = |b|$$

Hence by the maximum modulus principle, $h(w) \in \Re S_{n,m}^{q}(\lambda, b, \delta)$.

Hence the proof of Theorem 1 is completed.

Corollary B1 : $h(w)\Re S^{q}_{n,m}(\lambda, b, \delta)$ then

$$a_x \leq \frac{|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(x)}$$

where

$$Q(x) = \left(\frac{\lambda+q}{\lambda+x}\right) \begin{pmatrix} x \\ m \end{pmatrix} [\lambda(x-m-1)+\delta][x-q+|b|].$$

Theorem 2 : A function $h(w) \in A(n)$ and given by (1.1) is in $\Re G^{q}_{n,m}(\lambda, b, \delta)$ if and only if

$$\sum_{x=n+q}^{\infty} xQ(x)a_x \leq q|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]. \tag{2.4}$$

Proof : Let $h(w) \in \Re G^{q}_{n,m}(\lambda, b, \delta)$. Therefore

$$wh'(w) \in \Re S^{q}_{n,m}(\lambda, b, \delta).$$

Let $g(w) = wh'(w)$. Then $g(w) \in \Re S^{q}_{n,m}(\lambda, b, \delta)$. Therefore

$$\left| \frac{1}{b} \left(\frac{\delta w(\mathcal{F}^{\lambda}_{q} g(w))^{(m+1)} + \lambda w^2 (\mathcal{F}^{\lambda}_{q} g(w))^{(m+2)}}{\lambda w(\mathcal{F}^{\lambda}_{q} g(w))^{(m+1)} + (\delta-\lambda)(\mathcal{F}^{\lambda}_{q} g(w))^{(m)}} - (q-m) \right) \right| < 1 \tag{2.5}$$

$$\begin{aligned} &\delta w(\mathcal{F}^{\lambda}_{q} g(w))^{(m+1)} + \lambda w^2 (\mathcal{F}^{\lambda}_{q} g(w))^{(m+2)} \\ &= \begin{pmatrix} q \\ m \end{pmatrix} q(q-m)[\delta + \lambda(q-m-1)w^{q-m} \\ &- \sum_{x=n+q}^{\infty} \begin{pmatrix} x \\ m \end{pmatrix} \left(\frac{\lambda+q}{\lambda+x}\right) x(x-m)[\delta + \lambda(x-m-1)]a_x w^{x-m}. \end{aligned}$$

Now consider

$$\begin{aligned} &\lambda w(\mathcal{F}^{\lambda}_{q} g(w))^{(m+1)} + (\delta-\lambda)(\mathcal{F}^{\lambda}_{q} g(w))^{(m)} \\ &= q \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m)+\delta-\lambda]w^{q-m} \\ &- \sum_{x=n+q}^{\infty} \begin{pmatrix} x \\ m \end{pmatrix} \left(\frac{\lambda+q}{\lambda+x}\right) x[\lambda(x-m)+\delta-\lambda]a_x w^{x-m}. \end{aligned}$$

From (2.5) we have

$$\left| \frac{1}{b} \left(\frac{\sum_{x=n+q}^{\infty} \begin{pmatrix} x \\ m \end{pmatrix} \left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta][-x+q]a_x w^{x-m}}{\begin{pmatrix} q \\ m \end{pmatrix}[\lambda(q-m-1)+\delta]w^{q-m} - \sum_{x=n+q}^{\infty} \begin{pmatrix} x \\ m \end{pmatrix} \left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta]a_x w^{x-m}} \right) \right| < 1 \tag{2.6}$$

We know that $Re(w) < |w|, w \to 1_-$ through real axis, (2.6) leads us that

$$\frac{1}{|b|}\left[\frac{\sum_{x=n+q}^{\infty}\left(\begin{array}{c}x\\m\end{array}\right)\left(\frac{\lambda+q}{\lambda+x}\right)[\lambda(x-m-1)+\delta]x[x-q]a_x}{q\left(\begin{array}{c}q\\m\end{array}\right)[\lambda(q-m-1)+\delta]-\sum_{x=n+q}^{\infty}\left(\begin{array}{c}x\\m\end{array}\right)\left(\frac{\lambda+q}{\lambda+x}\right)x[\lambda(x-m-1)+\delta]a_x}\right] < 1.$$

On simplifying we get

$$\sum_{x=n+q}^{\infty} xQ(x)a_x \leq q|b|\left(\begin{array}{c}q\\m\end{array}\right)[\lambda(q-m-1)+\delta].$$

Hence we get (2.4).

Now we will prove the converse, from (2.4) and by setting $|w| = 1$ we see that

$$\left|\left(\frac{\delta w(\mathcal{F}_q^{\lambda}(g(w)))^{(m+1)}+\lambda w^2(\mathcal{F}_q^{\lambda}(g(w)))^{(m+2)}}{\lambda w(\mathcal{F}_q^{\lambda}(g(w)))^{(m+1)}+(\delta-\lambda)(\mathcal{F}_q^{\lambda}(g(w)))^{(m)}}-(-m)\right)\right|$$

$$=\left|\left(\frac{\sum_{x=n+q}^{\infty}\left(\begin{array}{c}x\\m\end{array}\right)\left(\frac{\lambda+q}{\lambda+x}\right)x[\lambda(x-m-1)+\delta][x-q]a_x w^{x-m}}{\left(\begin{array}{c}q\\m\end{array}\right)[\lambda(q-m-1)+\delta]w^{q-m}-\sum_{x=n+q}^{\infty}\left(\begin{array}{c}x\\m\end{array}\right)\left(\frac{\lambda+q}{\lambda+x}\right)x[\lambda(x-m-1)+\delta]a_x w^{x-m}}\right)\right|$$

$$\leq |b|\frac{q\left(\begin{array}{c}q\\m\end{array}\right)[\lambda(q-m-1)+\delta]w^{q-m}-\sum_{x=n+q}^{\infty}\left(\begin{array}{c}x\\m\end{array}\right)\left(\frac{\lambda+q}{\lambda+x}\right)x[\lambda(x-m-1)+\delta]a_x w^{x-m}}{q\left(\begin{array}{c}q\\m\end{array}\right)[\lambda(q-m-1)+\delta]w^{q-m}-\sum_{x=n+q}^{\infty}\left(\begin{array}{c}x\\m\end{array}\right)\left(\frac{\lambda+q}{\lambda+x}\right)x[\lambda(x-m-1)+\delta]a_x w^{x-m}} = |b|$$

Hence by the maximum modulus principle, $g(w) \in \Re S_{n,m}^{q}(\lambda, b, \delta)$.

Hence the proof of Theorem 2 is completed.

Corollary B2 : $h(w) \in \Re G_{n,m}^{q}(\lambda, b, \delta)$ then

$$a_x \leq \frac{q|b|\left(\begin{array}{c}q\\m\end{array}\right)[\lambda(q-m-1)+q]}{xQ(x)}, \quad x \geq n+q.$$

3. Growth and Distortion Theorem

Theorem 3 : If $h(w) \in \Re S_m^q(\lambda, b, \delta)$ then

$$|w|^q - |w|^{n+q} \frac{|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(n+q)} \leq |h(w)| \leq |w|^q + |w|^{n+q} \frac{|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(n+q)}.$$

With equality hold for

$$h(w) = w^q - w^{n+q} \frac{|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(n+q)}$$

Proof : $h(w) \in \Re S_{n,m}^q(\lambda, b, \delta)$. Therefore from (2.1)

$$\begin{aligned} \sum_{x=n+q}^{\infty} a_x &\leq \frac{|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(x)} \\ |h(w)| &\geq |w|^q - \sum_{x=n+q}^{\infty} |a_x||w|^x \geq |w|^q - |w|^{n+q} \sum_{x=n+q}^{\infty} |a_x| \\ &\geq |w|^q - |w|^{n+q} \frac{|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(n+q)}. \end{aligned}$$

Similarly

$$|h(w)| \leq |w|^q + |w|^{n+q} \frac{|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(n+q)}$$

Therefore we get required result.

Theorem 4 : If $h(w) \in \Re G_{n,m}^q(\lambda, b, \delta)$ then

$$|w|^q - |w|^{n+q} \frac{q|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{(n+q)Q(n+q)} \leq |h(w)| \leq |w|^q + |w|^{n-q} \frac{q|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{(n+q)Q(n+q)}.$$

With equality hold for

$$h(w)=w^q-w^{n+q}\frac{q|b|\begin{pmatrix} q \\ m \end{pmatrix}[\lambda(q-m-1)+\delta]}{(n+q)Q(n+q)}.$$

Proof : $h(w)\in \Re G^q_{n,m}(\lambda,b,\delta)$. Therefore

$$\sum_{x=n+q}^{\infty} a_x \le \frac{q|b|\begin{pmatrix} q \\ m \end{pmatrix}[\lambda(q-m-1)+\delta]}{xQ(x)}.$$

We have

$$|h(w)|\ge |w|^q-\sum_{x=n+q}^{\infty}|a_x||w|^x \ge |w|^q-|w|^{n+q}\sum_{x=n+q}^{\infty}|a_x|$$

$$\ge |w|^q-|w|^{n+q}\frac{q|b|\begin{pmatrix} q \\ m \end{pmatrix}[\lambda(q-m-1)+\delta]}{(n+q)Q(n+q)}.$$

Similarly

$$|h(w)|\le |w|^q+|w|^{n+q}\frac{q|b|\begin{pmatrix} q \\ m \end{pmatrix}[\lambda(q-m-1)+\delta]}{(n+q)Q(n+q)}.$$

Therefore we get required result.

Theorem 5 : If $h(w)\in \Re S^q_{n,m}(\lambda,b,\delta)$ then

$$q|w|^{q-1}+|w|^{n+q-1}\frac{|b|\begin{pmatrix} q \\ m \end{pmatrix}(n+q)[\lambda(q-m-1)+\delta]}{Q(n+q)}\le |h'(w)|$$

$$\le q|w|^{q-1}+|w|^{n+q-1}\frac{|b|\begin{pmatrix} q \\ m \end{pmatrix}(n+q)[\lambda(q-m-1)+\delta]}{Q(n+q)}.$$

Proof : $h(w)\in \Re S^q_{n,m}(\lambda,b,\delta)$. Therefore from (2.1)

$$\sum_{x=n+q}^{\infty} a_x \le \frac{|b|\begin{pmatrix} q \\ m \end{pmatrix}[\lambda(q-m-1)+\delta]}{Q(x)}$$

$$|h'(w)|\ge q|w|^{q-1}-\sum_{x=n+q}^{\infty}|a_x|x|w|^{x-1}\ge q|w|^{q-1}-|w|^{n+q-1}(n+q)\sum_{x=n+q}^{\infty}|a_x|$$

$$\ge q|w|^{q-1}-|w|^{n+q-1}\frac{|b|\begin{pmatrix} q \\ m \end{pmatrix}(n+q)[\lambda(q-m-1)+\delta]}{Q(n+q)}.$$

Similarly

$$|h'(w)| \leq q|w|^{q-1} + |w|^{n+q-1} \frac{|b| \begin{pmatrix} q \\ m \end{pmatrix} (n+q)[\lambda(q-m-1)+\delta]}{Q(n+q)}.$$

Therefore we get required result.

Theorem 6 : If $h(w) \in \Re G^q_{n,m}(\lambda, b, \delta)$ then

$$q|w|^{q-1} + |w|^{n+q-1} \frac{q|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(n+q)} \leq |h'(w)|$$
$$\leq q|w|^{q-1} + |w|^{n+q-1} \frac{q|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(n+q)}.$$

Proof : $h(w) \in \Re G^q_{n,m}(\lambda, b, \delta)$. Therefore from Theorem 1

$$\sum_{x=n+q}^{\infty} a_x \leq \frac{q|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{xQ(x)}$$
$$|h'(w)| \geq q|w|^{q-1} - \sum_{x=n+q}^{\infty} |a_x| x |w|^{x-1} \geq q|w|^{q-1} - |w|^{n+q-1}(n+q) \sum_{x=n+q}^{\infty} |a_x|$$
$$\geq q|w|^{q-1} - |w|^{n+q-1} \frac{q|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(n+q)}.$$

Similarly

$$|h'(w)| \leq q|w|^{q-1} + |w|^{n+q-1} \frac{q|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}{Q(n+q)}.$$

Therefore we get required result.

4. Radius of Convexity

Theorem 7 : If $h(w) \in \Re S^q_{n,m}(\lambda, b, \delta)$, then $h \in \mathcal{K}(\alpha)$ in $|w| < r_1(q, n, m, \lambda, b, \delta, \alpha)$ where

$$r_1 = \inf_x \left(\left(\frac{(q-\alpha)Q(x)}{x|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]} \right)^{\frac{1}{x-q}} \right).$$

Proof : We need to show that $\left|\frac{h'(w)}{w^{q-1}} - q\right| < q - \alpha$

$$\left|\frac{h'(w)}{w^{q-1}} - q\right| \leq \sum_{x=n+q}^{\infty} x|a_x||w|^{x-q} < q - \alpha. \tag{4.1}$$

From (2.1) we have

$$\sum_{x=n+q}^{\infty} a_x \leq \frac{|b|\binom{q}{m}[\lambda(q-m-1)+\delta]}{Q(x)}.$$

That is

$$\sum_{x=n+q}^{\infty} \frac{Q(x)}{|b|\binom{q}{m}[\lambda(q-m-1)+\delta]} a_x \leq 1. \tag{4.2}$$

Observe that (4.1) is true if

$$\frac{x|w|^{x-q}}{q-\alpha} \leq \frac{Q(x)}{|b|\binom{q}{m}[\lambda(q-m-1)+\delta]}.$$

Therefore

$$|w| \leq \left(\frac{(q-\alpha)Q(x)}{x|b|\binom{q}{m}[\lambda(q-m-1)+\delta]}\right)^{\frac{1}{x-q}},$$

$(q \neq x, q, x \in \mathbb{N})$, which complete the proof.

Theorem 8 : If $h(w) \in \Re S^{q}_{n,m}(\lambda, , b, \delta)$, then $h \in S^*(\alpha)$ in $|w| < r_2(q, n, m, \lambda, b, \delta, \alpha)$ where

$$r_2 = \inf_x \left(\left(\frac{(q-\alpha)Q(x)}{(x-\alpha)|b|\binom{q}{m}[\lambda(q-m-1)+\delta]}\right)^{\frac{1}{x-q}}\right).$$

Proof : We must show that

$$\left|\frac{wh'(w)}{h(w)} - q\right| < q - \alpha$$

We have

$$\left|\frac{wh'(w)}{h(w)} - q\right| = \left|\frac{-\sum_{x=n+q}^{\infty}(x-q)a_x w^x}{w^q - \sum_{x=n+q}^{\infty} a_x w^x}\right| \leq \frac{\sum_{x=n+q}^{\infty}(x-q)|a_x||w|^{x-q}}{1 - \sum_{x=n+q}^{\infty}|a_k||w|^{k-q}} \leq q - \alpha. \tag{4.3}$$

Hence (4.3) holds true if

$$\sum_{x=n+q}^{\infty} \frac{(x-\alpha)}{(q-\alpha)} |a_x||w|^{x-q} \leq 1. \tag{4.4}$$

From (2.1) we have

$$\sum_{x=n+q}^{\infty} \frac{Q(x)}{|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]} a_k \leq 1. \tag{4.5}$$

Hence by using (4.4) and (4.5) we get

$$\frac{(x-\alpha)}{(q-\alpha)} |w|^{x-q} \leq \frac{Q(x)}{|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}$$

$$|w|^{x-q} \leq \frac{(q-\alpha)Q(x)}{(x-\alpha)|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]}$$

$$|w| \leq \left(\frac{(q-\alpha)Q(x)}{(x-\alpha)|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]} \right)^{\frac{1}{x-q}}$$

$(q \neq x, q, x \in \mathbb{N})$, which complete the proof.

Theorem 9 : If $h(w) \in \Re S^q_{n,m}(\lambda, b, \delta)$, then $h \in C(\alpha)$ in $|w| < r_3(q, n, m, \lambda, b, \delta, \alpha)$ where

$$r_3 = \inf_x \left(\left(\frac{q(q-\alpha)Q(x)}{x(x-\alpha)|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]} \right)^{\frac{1}{x-q}} \right).$$

—noi **Proof** : h is convex if wh' is starlike. We must show that

$$\left| \frac{wg'(w)}{g(w)} - q \right| \leq q - \alpha$$

$$\sum_{x=n+q}^{\infty} \frac{x(x-\alpha)}{q(q-\alpha)} |a_x||w|^{x-q} \leq 1. \tag{4.6}$$

From (2.1) we have

$$\sum_{x=n+q}^{\infty} \frac{Q(x)}{|b| \begin{pmatrix} q \\ m \end{pmatrix} [\lambda(q-m-1)+\delta]} a_x \leq 1. \tag{4.7}$$

Hence by using (4.6) and (4.7) we get

$$\frac{x(x-\alpha)}{q(q-\alpha)}|w|^{x-q} \leq \frac{Q(x)}{|b|\begin{pmatrix} q \\ m \end{pmatrix}[\lambda(q-m-1)+\delta]}$$

$$|w| \leq \left(\frac{q(q-\alpha)Q(x)}{x(x-\alpha)|b|\begin{pmatrix} q \\ m \end{pmatrix}[\lambda(q-m-1)+\delta]} \right)^{\frac{1}{x-q}}$$

$(q \neq x, q, x \in \mathbb{N})$, which complete the proof.

5. Closure Theorem

Theorem 10 : Let $h_1(w) = w^q$ and

$$h_x(w) = w^q - \frac{Q(x)}{|b|\begin{pmatrix} q \\ m \end{pmatrix}[\lambda(q-m-1)+\delta]} w^x \quad \text{for} \quad x \geq n+q.$$

Then $h(w) \in \Re S^q_{n,m}(\lambda, b, \delta)$ if and only if $h(w)$ can be expressed in the form

$$h(w) = \lambda_1 h_1(w) + \sum_{x=n+q}^{\infty} \lambda_x h_x(w)$$

where $\lambda_x \geq 0$ and $\lambda_1 + \sum\limits_{x=n+q}^{\infty} \lambda_x = 1$.

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