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DERIVATIONS IN PRIME NEAR RINGS

A. R. GOTMARE
GDM Arts KRN Commerce and MD Science College,
Jamner. Dist Jalgaon, India
E-mail: gotmarear@gmail.com

Abstract

In this paper I have proved following two results for N be a prime near rings.

(1) : If there exists $k, l \in \square$ such that N admits a derivation d satisfying $d([x,y]) = x^k[x,y]x^l$ for $x, y \in N$ or $d([x,y]) = -x^k[x,y]x^l$ for all $x, y \in N$ then N is commutative ring.

(2) If there exists $k, l \in \square$ such that admits a derivation d satisfying

$$d(x \circ y) = x^k(x \circ y)x^l$$

for $x, y \in N$ or $d(x \circ y) = -x^k(x \circ y)x^l$ for $x, y \in N$ then N is a commutative ring.

1. Introduction

Throughout this paper N stands for non zero symmetric right near ring.

N is called non zero symmetric ring if $x_0 = 0$ for all $x \in N$. According to [4], a near ring N is said to be prime if $xNx = \{0\}$, $\forall\ x, y \in N \Rightarrow x = 0$, or $y = 0$. An additive endomorphism d of N is called a derivation on N if $d(xy) = xd(y) + d(x)y$, $\forall x, y \in N$ or equivalently. As noted in [5] as $d(xy) = d(x)y + xd(y), \forall\ x, y \in N$. Let $Z(N)$ be the multiplicative centre of N i.e. $Z(N) = \{x \in N/xy = yx,\ \forall\ y \in N\}$. Note that $\square(N)$ is not empty. Now for any $x, y \in N$ we denote $[x,y] = xy - yx$and $x \circ y = xy + yx$, the

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well know Lie and Jordan Product respectively.

In [5] the authors have showed that a prime ring R must be commutative if it admits a derivation d such that $d([x,y]) = [x,y]$ for all $x,y \in K$ or $d([x,y]) = -[x,y]$ for all $x,y \in K$, where K is non zero ideal of R. The main purpose of this paper is to go further step. In this direction and extend the result on prime rings admitting special type of generalized derivation to prime near rings.

2. The Results

Let $\Box$ be non negative integers including 0. The proof of our main result depends on the following lemma.

Lemma 2.1 : Let N be a prime near ring. If N admits a non zero derivation d for which $d(N) \subset Z(N)$ then N is commutative.

Theorem 2.2 : Let N be a prime near ring. If there exists $k,l \in \Box$ such that N admits a non zero derivation d satisfying

(i) $d([x,y]) = x^k[x,y]x^l, \quad \forall \quad x,y \in N$

(ii) $d([x,y]) = -x^k[x,y]x^l, \ \forall \ x,y \in N$ then N is a commutative ring.

Proof : Suppose that (i) holds.

Since $[x,yx] = [x,y]x$, replacing y by yx in (i), we obtain

$$d([x,y]x) = d([x,yx]) = x^k[x,yx]x^l = x^k[x,y]x^{l+1} \text{ for all } x,y \in N. \tag{1.1}$$

By definition, we have $d([x,y]x) = d([x,y])x + [x,y]d(x)$.

Using (1.1) and assumption (i) we have

$$x^k[x,y]x^{l+1} = x^k[x,y]x^{l+1} + [x,y]d(x). \tag{1.2}$$

And thus $[x,y]d(x) = 0$. Now replacing y by zy, we have

$$[x,zy]d(x) = [x,z]yd(x) = 0 \text{ for all } x,y,z \in N \tag{1.3}$$

which implies

$$[x,z]Nd(x) = 0 \text{ for all } x,z \in N. \tag{1.4}$$

In view of primeness (1.4) yields that for each $x \in N$

$$\Rightarrow d(x) = 0 \text{ or } x \in Z(N). \tag{1.5}$$

We know that if $x \in Z(N)$ then $d(x) \in Z(N)$.

Hence (1.5) forces that $\forall\ x \in N, d(x) \in Z(N)$. This follows Lemma 2.1 that N Is a commutative ring.

Now suppose (ii) holds.

Substituting yx for y in (ii). We can similarly obtain

$$d([x,y]x) = d([x,yx]) = -x^k[x,yx]x^l = -x^k[x,y]x^{l+1} \text{ for all } x, y \in N. \tag{1.6}$$

By definition, we have $([x,y],x) = d([x,y])x + [x,y]d(x)$. Using (1.6) and assumption (ii) we have

$$-x^k[x,y]x^{l+1} = -x^k[x,y]x^{l+1} + [x,y]d(x). \tag{1.7}$$

And thus $[x,y]d(x) = 0$. □

The following simple example says the primeness hypothesis in Theorem 2.2 is necessary even in in the case of arbitrary rings.

Example 2.3 : Let R be a non trivial commutative ring and $N = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$ such that $x, y \in R$. If we define $d : N \to N$ by $d\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, then d is non zero derivation on N.

If $A = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, where $x \neq 0$ then $ANA = 0$ which means N is not prime. Further d satisfies the condition $d([A,B]) = [A,B]$ for all $A, B \in N$, but N is non commutative ring.

Theorem 2.4 : Let N be a prime near ring. If there exists $k, l \in \square$ such that N admits a non zero derivation d satisfying either

(i) $d(x \circ y) = x^k(x \circ y)x^l$ OR

(ii) $d(x \circ y) = -x^k(x \circ y)x^l$ for all $x, y \in N$.

Then N is a commutative ring.

Proof : We first assume that (i) holds.

Since $(y \circ yx) = (x \circ y)x$, and replacing y by yx in equation (i), we have

$$d((x \circ y)x) = d(x \circ (yx)) = x^k(x \circ (yx))x^l = x^k(x \circ y)x^{l+1} \text{ for all } x, y \in N. \tag{1.8}$$

By definition, we have $d((x \circ y)x) = d(x \circ y)x + (x \circ y)x$.

Then by assumption (i) we have

$$d((x \circ y)x) = x^l(x \circ y)x^{k+1} + (x \circ y)d(x). \tag{1.9}$$

Combining (1.8) and (1.9) we conclude that

$$(x \circ y)d(x) = 0 \tag{1.10}$$

We obtain

$$xzyd(x) = -zyxd(x) = (-z)(-xyd(x)) - (-z)(-x)yd(x). \tag{1.11}$$

And thus $(xz - (-z)(-x)y)d(x) = 0$ for all $x, y, z \in N$. Replacing x by $-x$ gives

$$(-xz + zx)yd(-x) = 0 \tag{1.12}$$

And hence

$$-[z, x]yd(x) = [z, x]yd(-x) = 0 \quad \text{for all} \quad x, y, z \in N. \tag{1.13}$$

By virtue of primness of N, we have for each $x \in N$

$$d(x) = 0 \quad \text{or} \quad x \in Z(N). \tag{1.14}$$

Since (1.14) is the same as (1.5) and arguing as the proof of Theorem 2.2 we obtain that N is a commutative ring.

Next, we assume that (ii) holds. Substituting yx for y in (ii) we can similarly obtain

$$d((x \circ y)x) = d(x \circ (yx)) = -x^k(x \circ (yx))x^l = -x^k(x \circ y)x^{l+1} \quad \text{for all} \quad x, y \in N. \tag{1.15}$$

By definition, we have

$$d((x \circ y)x) = -x^2(x \circ y)x^{l+1} + (x \circ y)d(x). \tag{1.16}$$

Combining (1.15) and (1.16) we conclude that $(x \circ y)d(x) = 0$. The rest of the proof is same as before. □

The following simple example demonstrate that the prime ness hypothesis in Theorem 2.4 is necessary even in the case of arbitrary rings.

Example : Let S be a ring and $N = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & z \\ y & 0 & 0 \end{pmatrix}$ such that $x, y, z \in s$ and define a map $d : N \to N$ such that

$$d\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & z \\ y & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$$

Clearly, the endomorphism d is a derivation on N. If $A = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $x \neq 0$, then $ANA = 0$ which proves that N is not prime. Further more, d satisfies the condition $d(A \circ B) = A \circ B$ for all $A, B \in N$, but N is not commutative ring.

2. Conclusion

In this paper we study the prime near ring with derivations. We prove that a prime near ring which admits a non zero derivation satisfying certain differential identities is commutative ring.

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