International J. of Pure & Engg. Mathematics (IJPEM)
ISSN 2348-3881, Vol. 4 No. II (August, 2016), pp. 1-9

STRONGLY MAGIC SQUARE AS VECTOR SPACE

NEERADHA C. K.[1] AND V. MADHUKAR MALLAYYA[2]

[1] Assistant Professor, Dept. of Science & Humanities,
Mar Baselios College of Engineering & Technology,
Thiruvananthapuram, India
E-mail: ckneeradha@yahoo.co.in

[2] Professor & HOD, Dept. of Mathematics, Mohandas College of Engineering & Technology,
Thiruvananthapuram, India
E-mail: mmallayyav@gmail.com

Abstract

A magic square is a square array of numbers where the rows, columns, diagonals and co-diagonals add up to the same number. The paper discuss about a well-known class of magic square; the strongly magic square. A generic definition for strongly magic square is given and advanced mathematical structure of strongly magic squares is discussed.In this paper strongly magic squares are proved to be an abelian group and a vectorspace.

1. Introduction

A normal magic square of order n is a square array of consecutive numbers from $1 \cdots n^2$ where the rows, columns, diagonals and co-diagonals add up to the same number. The

---

Key Words : *Magic Square, Magic Constant, Strongly magic square, Abelian group,Vector space.*

AMS Subject Classification : 15A03.

constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, strongly magic square will have a stronger property that the sum of the entries of the sub squares taken without any gaps between the rows or columns is also the magic constant. For example, strongly magic square of order 4 with an additional property that the sum of the entries of the 2×2 subsquares taken without any gaps between the rows or columns is the magic constant [2].

There are many recreational aspects of strongly magic squares. But, apart from the usual recreational aspects, it is found that these strongly magic squares possess advanced mathematical structures.

2. Notations and Mathematical Preliminaries

A. Magic Square

A magic square of order n is an n^{th} order matrix $[a_{ij}]$ such that

$$\sum_{j=1}^{n} a_{ij} = \rho \quad \text{for} \quad i = 1, 2, \cdots, n \tag{1}$$

$$\sum_{j=1}^{n} a_{ji} = \rho \quad \text{for} \quad i = 1, 2, \cdots, n \tag{2}$$

$$\sum_{i=1}^{n} a_{ii} = \rho \qquad \sum_{i=1}^{n} a_{i,n-i+1} = \rho. \tag{3}$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and ρ symbol represents the magic constant. [3]

B. Magic Constant

The constant ρ in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$.

C. Strongly magic square (SMS): Generic Definition

Let $A = [a_{ij}]$ be a matrix of order $n^2 \times n^2$, such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \quad \text{for} \quad i = 1, 2, \cdots, n^2 \tag{4}$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \quad \text{for} \quad i = 1, 2, \cdots, n^2 \tag{5}$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho, \quad \sum_{i=1}^{n^2} a_{i,n^2-i+1} = \rho \tag{6}$$

$$\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} a_{i+k,j+l} = \rho \text{ for } i,j = 1,2,\cdots,n^2. \tag{7}$$

where the subscripts are congruent modulo n^2.

Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal and co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns and is denoted as $M_{OC}^{(n)}$ or $M_{OR}^{(n)}$ and ρ is the magic constant.

Note : The n^{th} order sub square sum with k column gaps or k row gaps is generally denoted as $M_{kC}^{(n)}$ or $M_{kR}^{(n)}$ respectively.

D. Group

A group $(G, *)$ is a nonempty set G closed under a binary operation $*$ such that the following axioms are satisfied

(i) $*$ is associative in G. i.e, $a*(b*c) = (a*b)*c, \quad \forall \ a,b,c \in G$

(ii) $\exists \ e \in G$, such that $e*a = a*e, \ \forall \ a \in G$, where e is the identity element for $*$.

(iii) Corresponding to each $a \in G, \ \exists \ b \in G$ such that $a*b = b*a = e$, where b is the inverse of a[4, 5].

E. Abelian Group

A group G is abelian if its binary operation $*$ is commutative. [4]

F. Vector Space

A non-empty set together with two operations $+$ and $\cdot$ called addition and scalar multiplication respectively, is called a vector space or linear space over a field F if the following conditions are satisfied.

(i) $\langle V, + \rangle$ is an abelian group.

(ii) $\forall \ \lambda \in F$ and $a \in V, \ \lambda \cdot a \in V$

(iii) $\forall \ \lambda \in F$ and $a, b \in V, \lambda(a+b) = \lambda \cdot a + \lambda \cdot b$

(iv) $\forall \ \lambda, \mu \in F$ and $a \in V, (\lambda + \mu) \cdot a = \lambda \cdot a + \mu \cdot a$

(v) $\forall\ \lambda, \mu \in F$ and $a \in V, (\lambda\mu) \cdot a = \lambda \cdot (\mu a)$

(vi) $1 \cdot a = a,\ \forall\ a \in V$ and 1 is the unity element of the field F[6].

G. Other Notations

1. SM_s denote the set of all strongly magic squares
2. $SM_{S(a)}$ denote the set of all strongly magic squares of the form $[a_{ij}]_{n^2\times n^2}$ such that $a_{ij} = a$ for every $i, j = 1, 2, \cdots, n^2$. Here A is denoted as $[a]$, i.e., if $A \in SM_{S(a)}$ then $\rho(A) = n^2 a$.
3. $SM_{S(0)}$ denote the set of all strongly magic squares with magic constant 0, i.e., if $A \in SM_{S(0)}$ then $\rho(A) = 0$.

3. Propositions and Theorems

Proposition 1 : If A and B are two SMSs of order $n^2 \times n^2$ with $\rho(A) = a$ and $\rho(B) = b$, then $C = (\lambda + \mu)(A + B)$ is also a SMS with magic constant $(\lambda + \mu)(\rho(A) + \rho(B))$; for every $\lambda, \mu \in R$.

Proof : Let $A = [a_{ij}]_{n^2\times n^2}$ and $B = [b_{ij}]_{n^2\times n^2}$. Then

$$\begin{aligned} C &= (\lambda + \mu)(A + B) \\ &= (\lambda + \mu)[a_{ij} + b_{ij}] \\ &= [(\lambda + \mu)(a_{ij} + b_{ij}]. \end{aligned}$$

Sum of the i^{th} row elements of

$$\begin{aligned} C &= \sum_{j=1}^{n^2} c_{ij} \\ &= \sum_{j=1}^{n^2} ((\lambda + \mu)(a_{ij} + b_{ij})) \\ &= (\lambda + \mu) \left(\sum_{j=1}^{n^2} (a_{ij}) + \sum_{j=1}^{n^2} (b_{ij}) \right) \\ &= (\lambda + \mu)(a + b) \\ &= (\lambda + \mu)(\rho(A) + \rho(B)). \end{aligned}$$

A similar computation holds for column sum.

Main diagonal sum

$$\begin{aligned}\sum_{i=1}^{n^2} c_{ii} &= \sum_{i=1}^{n^2}[(\lambda+\mu)(a_{ii}+b_{ii})] \\ &= (\lambda+\mu)\left(\sum_{i=1}^{n^2}(a_{ii})+\sum_{i=1}^{n^2}(b_{ii})\right) \\ &= (\lambda+\mu)(a+b) \\ &= (\lambda+\mu)(\rho(A)+\rho(B)).\end{aligned}$$

A similar computation holds for co - diagonal sum.

The sum of the $n \times n$ sub squares $M_{kC}^{(n)}$ is given by

$$\begin{aligned}\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} c_{i+k,j+l} &= \sum_{l=0}^{n-1}\sum_{k=0}^{n-1}(\lambda+\mu)(a_{i+k,j+l}+b_{i+k,j+l}) \\ &= (\lambda+\mu)\left(\sum_{l=0}^{n-1}\sum_{k=0}^{n-1}(a_{i+k,j+l}+\sum_{l=0}^{n-1}\sum_{k=0}^{n-1}(b_{i+k,j+l})\right) \\ &= (\lambda+\mu)(a+b) \\ &= (\lambda+\mu)(\rho(A)+\rho(B)).\end{aligned}$$

From the above propositions the following results can be obtained.

Results

If for every $\lambda, \mu \in R$ and $A, B \in SM_S$,

(1) $\lambda(A+B) \in SM_S$ with $\rho(\lambda(A+B)) = \lambda(\rho(A)+\rho(B))$.

Proof : By putting $\mu = 0$ in Proposition 1, result can be obtained.

(2) $(A+B) \in SM_S$ with $\rho((A+B)) = (\rho(A)+\rho(B))$.

Proof : It can be deduced by putting $\lambda = 1$ in Result 1.

(3) $\lambda A \in SM_S$ with $\rho(\lambda A) = \lambda\rho(A)$.

Proof : It can be easily verified by putting $B = 0$ in Result 1.

(4) $(\lambda+\mu)(A) \in SM_S$ with $\rho(\lambda+\mu)(A) = (\lambda+\mu)\rho(A)$.

Proof : In the Proposition 1 put $B = 0$, where $0 \in SM_S$.

(5) $\lambda A + \mu B \in SM_S$ with $\rho(\lambda A + \mu B) = \lambda\rho(A) + \mu\rho(B)$.

Proof ; It can be deduced from Result 2 and 3.

(6) $-A \in SM_S$ with $\rho(-A) = -\rho(A)$.

Proof ; In the above result 3 put $\lambda = -1$.

(7) $A - B \in SM_S$ with $\rho(A - B) = \rho(A) - \rho(B)$.

Proof : From the above result 2 and 6 it can be deduced

Theorem 1 : $\langle SM_S, +\rangle$ forms an abelin group.
Proof :

(I) Closure property : if $A, B, \in SM_S$, then $A + B \in SM_S$.(from above result 2).

(II) Associativity : if $A, B, C \in SM_S$, then $A+(B+C) = (A+B)+C \in SM_S$. (Since matrix addition is associative.)

(III) Existence of Identity: There exists 0 matrix in SM_S so that $A + 0 = 0 + A = A$, were 0 acts as the identity element.

(IV) Existence of additive inverse : For every $A \in SM_S$, there exists $-A \in SM_S$ so that $A + (-A) = 0$ where $0 \in SM_S$(from result 5).

(V) Commutativity : If $A, B \in SM_S$, then $A + B = B + A \in SM_S$ (Since matrix addition is commutative).

Proposition 2 : For all $A, B \in SM_S, \lambda, \mu \in R$;

(i) $\lambda(A + B) = \lambda A + \lambda B$

(ii) $(\lambda + \mu) \cdot A = \lambda \cdot A + \mu \cdot A$

(iii) $(\lambda\mu) \cdot A = \lambda \cdot (\mu \cdot A)$

(iv) $1 \cdot A = A$.

Proof : Since $A, B \in SM_S; A = [a_{ij}]_{n^2\times n^2}$ and $B = [b_{ij}]_{n^2\times n^2}$

(I) $A + B = [a_{ij} + b_{ij}]$

$$\begin{aligned}\lambda(A+B) &= \lambda[a_{ij}+b_{ij}]\\ &= [\lambda a_{ij}+\lambda b_{ij}]\\ &= [\lambda a_{ij}]+[\lambda b_{ij}]\\ &= \lambda[a_{ij}]+\lambda[b_{ij}]\\ &= \lambda\cdot A+\lambda\cdot B.\end{aligned}$$

(II)

$$\begin{aligned}(\lambda+\mu)\cdot A &= (\lambda+\mu)\cdot[a_{ij}]\\ &= [(\lambda+\mu)a_{ij}]\\ &= [\lambda a_{ij}+\mu a_{ij}]\\ &= [\lambda a_{ij}]+[\mu a_{ij}]\\ &= \lambda\cdot[a_{ij}]+\mu\cdot[a_{ij}]\\ &= \lambda\cdot A+\mu\cdot A.\end{aligned}$$

(III)

$$\begin{aligned}(\lambda\mu)\cdot A &= (\lambda\mu)\cdot[a_{ij}]\\ &= [\lambda\mu(a_{ij})]\\ &= \lambda[\mu a_{ij}]\\ &= \lambda\cdot(\mu\cdot A).\end{aligned}$$

(IV) $1\cdot A = 1\cdot[a_{ij}] = [1\cdot a_{ij}] = [a_{ij}] = A.$

Theorem 2 : $\langle SM_S, +, \cdot\rangle$ forms a vector space over the field of real numbers.

Proof : It is an immediate consequence of Theorem 1 and Proposition 2.

Theorem 3 : $\langle SM_{S(a)}, +, \cdot\rangle$ forms a vector space over the field of real numbers.

Proof : Since $SM_{S(a)} C SM_S$ where C denotes the subset and SM_S is a vector space over the field of real numbers R with respect to the addition of matrices as addition of vectors and multiplication of a matrix by a scalar as scalar multiplication, it is enough

to show that $SM_{S(a)}$ is a subspace of SM_S. This can be verified by the fact; for every $\lambda, \mu \in R$ and $A, B \in SM_{S(a)}; \lambda A + \mu B \in SM_{S(a)}$.
Since $A, B \in SM_{S(a)}, A = [a]$ and $B = [b]$

$$\begin{aligned} \lambda A + \mu B &= \lambda[a] + \mu[b] \\ &= [\lambda a] + [\mu b] \\ &= [\lambda a + \mu b] \in SM_{S(a)}. \end{aligned}$$

Theorem 4 : $\langle SM_{S(0)}, +, \cdot \rangle$ forms a vector space over the field of real numbers.
Proof : Proceeding as above it is enough to show that for every $\lambda, \mu \in R$ and $A, B \in SM_{S(0)}; \lambda A + \mu B \in SM_{S(0)}$.
Since $A, B \in SM_{S(0)}; \rho(A) = 0$ and $\rho(B) = 0$.
Now $\rho(\lambda A + \mu B) = \lambda \rho(A) + \mu \rho(B)$ (From result 5) $= \lambda \cdot 0 + \mu \cdot 0 = 0$.
Thus $\lambda A + \mu B \in SM_{S(0)}$.

4. Conclusion

While magic squares are recreational in grade school, they may be treated somewhat more seriously in different linear algebra courses. The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding strongly magic squares are described. Several applications of the results regarding strongly magic squares can be explored.

Acknolwledgement

We express our sincere gratitude for the valuable suggestions given by Dr. Ramaswamy Iyer, Former Professor in Chemistry, Mar Ivanios College, Trivandrum in preparing this paper.

References

[1] Andrews W. S., Magic Squares and Cubes, 2nd rev. ed. New York: Dover, (1960).

[2] Padmakumar T. V., Strongly Magic Square, (April 1995).

[3] Charles Small, Magic Squares over fields, The American Mathematical Monthly, 95(7) (Aug.-Sep., 1988), 621-625.

[4] John B. Fraleigh, A first Course in Abstract Algebra, Seventh edition, Narosa Publishing House, New Delhi, (2003).

[5] Vasishta A. R. and Vasishtha A. K., Modern Algebra, Fifttieth edition, Krishna Prakashan Media (P)Ltd, Meerut, (2006).

[6] Sharma R. D., Theory and Problems of Linear Algebra,I. K. International Publishing House(P)Ltd, New Delhi, (2011).