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A MEMBERSHIP FUNCTION SOLUTION TO BATCH ARRIVAL QUEUES WITH VARYING FUZZY FUZZY BATCH SIZES AND VACATION POLICIES

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Abstract

This work construct the membership function of the system characteristic of batch arrival queuing system with vacation policies where the arrival batch size, arrival rate, service rate and server vacation rates are fuzzy numbers. The basic idea is to transform fuzzy queue with batch arrival queue into a family of conventional crisp queue using -cut approach. By using Zadch's extension principle, a pair of parametric nontimear programmes are developed to describe the family of crisp queues with batch arrival and vacationing server. Two numerical examples are solved successfully to illustrate the validity of proposed approach. Because the system characteristics are expressed by the membership functions, more information is provided for use by management. By extending this model to the fuzzy environment, fuzzy queues with a vacationing server are represented more accurately and the analytical results are more useful for system designers and practitioners.

1. Introduction

Queuing system in which server leaves for a vacation of random length have received

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Key Words : *Parametric nontimear programming, α-cut, Zadeh's principle, Single vacation, Multiple vacation.*

considerable attention in literature. Queuing systems with server vacation have attracted much attention from numerous researchers since Levy and Yechiali [1]. Server vacations are useful for the system in which the server wants to utilize his idle time for different purposes. An excellent survey of queuing systems with server vacations was found in Doshi [2] and Takasi [3] - and included numerous applications in the study of maintenance problems in production/inventory schedules, computer networks and digital communication systems. The generality and flexibility of these vacation models are useful in modeling many real life situations (see [4]). For example, in most computer systems, the processor is shared among various types of jobs and hence is not available to each type at all times. From the perspective of one job type, the processor alternates between handling its job type and other job types. To reflect the occasional unavailability of the processor in queuing systems, the server is modeled as taking vacations (See [5]). The $M^{[x]}/G/1$ queuing system with multiple vacation, was first studied by Baba [6]. He derived the expected queue length, waiting time and busy period distributions through a supplementary variable technique. Lee and Srinivasan [7] examined the control operating policy of Baba's [6] model using a general approach and presented applications in production/inventory systems and other areas. Lee et al. [9,10] further analyzed Lee and Srinivasan's model with a single vacation and multiple vacations, respectively. They also provided a probabilistic interpretation of the single (multiple) vacation system with a threshold policy.

In the literature described above, the batch inter-arrival times, customer service times and server vacation times are required to follow certain probability distributions. However, in many-real world applications, the parameter distributions may only be characterized subjectively; that is, the arrival, service and vacation patterns are typically described in everyday language summaries of central tendency, such as "the mean arrival rate is around 5 per day", "the mean service rate is about 10 per hour" or "the mean vacation rate is approximately 2 per week", rather than with complete probability distributions.

On the basis of Zadeh's extension principle [12,20] the possibility concept and fuzzy Markov Chains [13], Li and Lee [14] have derived analytical solutions for two fuzzy queues, namely, M/F/1 and FM/FM/1, where F donates fuzzy time and FM denotes fuzzified time. However as commanded by Negi and Lee [15] their approach is very complicated and is generally unsuitable to computational purposes and they propose the α-cut and two variable simulation [16] approach to analyze fuzzy queues. Unfortunately, their approach only provides crisp solutions; in other words, the membership function of the performance measures are not completely described. If we can derive the membership of some performance measure, we obtain a more reasonable and realistic performance measure because it maintains the fuzziness of input informations that can be used to represent the fuzzy system more accurately.

Kao et al., [11], therefore, adopt parametric programming to construct the membership functions of the performance measure for fuzzy queues, and successfully applied to four simple fuzzy queues with are or two fuzzy variables, namely M/F/1, F/M/1, F/F/1 and FM/FM/1. It seem that their approach is applicable to the fuzzy batch arrival queues. However, since the batch arrival queuing systems are much more complicated than the above four fuzzy queues, the solution procedure for batch arrival queue is not explicitly known and deserves further investigation.

All previous research on fuzzy queuing models is focused on ordinary queues with one or two fuzzy variables. In this paper, we develop an approach that provides system characteristic for queues with a vacationing server and four fuzzy variables: fuzzified batch arrival size, exponential arrival, service and vacation rates. Through α-cuts and Zadeh's extension principle, we transform the fuzzy queues to a family of crisp queues. As varies, the family of crisp queues is described and solved using parametric nontimear programming (NLP). The NLP solutions completely and successfully yield the membership functions of the system characteristic, including the expected number of customers in queue, waiting time in the system and the expected length of time the server is idle and busy.

The remainder of this paper is organized as follows. Section 2 presents the system characteristics of standard and fuzzy queuing models with a vacationing server. In section 3, a mathematical programming approach is developed to derive the membership functions of these system characteristics. To demonstrate the validity of the proposed approach, two realistic numerical examples are described and solved. Discussion is provided in section 4, and conclusions are drawn in section 5. For notational convenience, our model in this paper is hereafter denoted FM[x]/FM/1/FV, where FV represents the fuzzified exponential vacation rate.

2. Fuzzy Queues With a Vacationing Server

We consider a queuing system with two different policies. Under policy I (multiple vacations), the server takes consecutive vacations when there are no customers queued for service, while under policy II (single vacation), the server takes only a single vacation. It is assumed that customers arrive in batches according to a compound Poisson process with group arrival rate λ. Each batch size K of arrival is represented by trapezoidal fuzzy number. Using α-cuts, the trapezoidal arrival batch size can be represented by different levels of interval of confidences. Let this interval of confidence be represented by $[t_{1\alpha}, t_{2\alpha}]$. Since probability distributions for the α-cut sets can be represented by uniform distributions, we have

$$P(t_\alpha) = \frac{1}{t_{2\alpha} - t_{1\alpha}}, \quad t_{1\alpha} \le t_\alpha \le t_{2\alpha}. \tag{1}$$

Thus the mean of the distributing

$$E(T_\alpha) = \int_{t_{1\alpha}}^{t_{2\alpha}} \frac{1}{t_{2\alpha} - t_{1\alpha}} \quad t_\alpha dt_\alpha = \frac{1}{2}(t_{2\alpha} + t_{1\alpha}). \tag{2}$$

Similarly the second moment, we have

$$E(T^2\alpha) = \int_{t_{1\alpha}}^{t_{2\alpha}} \frac{1}{t_{2\alpha} - t_{1\alpha}} \quad t_\alpha^2 dt_\alpha = \frac{t_{2\alpha}^3 - t_{1\alpha}^3}{3(t_{2\alpha} - t_{1\alpha})}. \tag{3}$$

Using the well-known formula

$$Var(T_\alpha) = E(T_\alpha^2) - [E(T_\alpha)]^2,$$

the variance can now be obtained as

$$Var(T_\alpha) = \frac{1}{12}(t_{2\alpha} - t_{1\alpha})^2. \tag{4}$$

Customers arriving in batches at the server form a single-file queue and are served in order. The service time is exponentially distributed with rate μ. When there are no customers in the queue, the server takes a vacation with vacation length exponentially distributed with rate θ. Define L_q, the expected number of customers in queue, ws, the expected waiting time in the system. $E[I]$, the expected length of time the server is idle including vacations. $E[B]$, the expected length of time the server is busy.

From the results in Takaki [3] and Choudhury [17], we can easily derive the system characteristics for policies I and II in terms of the system parameters.

Policy I :

$$L_q = \left[\frac{\rho}{\mu(1-\rho)} + \frac{E[A(A-1)]}{2\mu E(A)(1-\rho)} + \frac{1}{\theta}\right]\lambda \tag{5}$$

$$W_s = \frac{1}{\lambda}\left[\frac{\theta^2}{(1-\rho)} + \frac{\lambda E[A(A-1)]}{2\mu(1-\rho)} + \lambda E[A]\left(\frac{1}{\theta}+\frac{1}{\mu}\right)\right] \tag{6}$$

$$E[I] = \frac{1}{\lambda} + \frac{1}{\theta} \tag{7}$$

$$E[B] = \frac{\rho}{(1-\rho)}\left(\frac{1}{\lambda}+\frac{1}{\theta}\right) \tag{8}$$

Policy II :

$$L_q = \lambda\left[\frac{\rho}{\mu(1-\rho)} + \frac{E[A(A-1)]}{2\mu E(A)(1-\rho)} + \frac{\lambda(\lambda+\theta)}{\theta(\theta^2+\lambda\theta+\lambda^2)}\right]\lambda \tag{9}$$

$$W_s = \frac{1}{\lambda}\left[\frac{\theta^2}{(1-\rho)} + \frac{\lambda E[A(A-1)]}{2\mu(1-\rho)} + \frac{\lambda^2 E[A](\lambda+\theta)}{\theta(\theta^2+\lambda\theta+\lambda^2)} + \frac{\lambda E(A)}{\mu}\right] \tag{10}$$

$$E[I] = \frac{\theta}{\lambda(\lambda+\theta)} + \frac{1}{\theta} \tag{11}$$

$$E[B] = \frac{\rho}{(1-\rho)}\left(\frac{1}{\lambda}+\frac{1}{\theta}\right) \tag{12}$$

where $\rho = \frac{\lambda E(A)}{\mu}$. In steady-state, it is necessary that we have $0 < \frac{\theta(1-\rho)}{\lambda+\theta} < 1$ and $0 < \frac{\theta(\lambda+\theta)(1-\rho)}{\lambda(\lambda+\theta)+\theta^2} < 1$ for policies I and II, respectively.

2.2. $FM^{[x]}/FM/1/FV$ Queues

To extend the applicability of the standard queuing model with a vacationing server, we allow for specification of system parameters. Suppose the batch arrival size w, the arrival rate λ, service rate μ and vacation rate θ are approximately known and can be represented by the fuzzy sets $\tilde{w}, \tilde{\lambda}, \tilde{\mu}$ and $\tilde{\theta}$ respectively. Let $\eta_{\tilde{w}}(a), \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y)$ and $\eta_{\tilde{\theta}}(v)$ denote the membership functions of $\tilde{w}, \tilde{\lambda}, \tilde{\mu}$ and $\tilde{\theta}$ respectively. We then have the following fuzzy sets:

$$\tilde{w} = \{(a, \eta_{\tilde{w}}(a))/a \in A\} \tag{13a}$$

$$\tilde{\lambda} = \{(x, \eta_{\tilde{\lambda}}(a))/x \in X\} \tag{13b}$$

$$\tilde{\mu} = \{(y, \eta_{\tilde{\mu}}(a))/y \in Y\} \tag{13c}$$

$$\tilde{\theta} = \{(v, \eta_{\tilde{\theta}}(a))/v \in V\} \tag{13d}$$

where A, X, Y and V are the crisp universal sets of batch size, arrival, service and vacation rates, respectively.

Let $f(a, x, y, v)$ denote the system characteristic of interest. Since $\tilde{A}, = tilde\lambda, \tilde{\mu}$ and $\tilde{\theta}$ are fuzzy numbers,$f(\tilde{A}, \tilde{\lambda}, \tilde{\mu}, \tilde{\theta})$ is also a fuzzy number. Following Zadch's extension principle (sec [21]). The membership function of the system characteristic $f(\tilde{A}, \tilde{\lambda}, \tilde{\mu}, \tilde{\theta})$ is defined as

$$\eta_{f(\tilde{A},\tilde{\lambda},\tilde{\mu},\tilde{\theta})}(z) = \sup_{\Omega} \min\{\eta_{\tilde{A}}(a), \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{\theta}}(v)/z = f(a, x, y, z)\} \tag{14}$$

where the supremum is taken over the set

$$\Omega = \left\{ a \in A, x \in X, y \in Y, v \in V/0 < \frac{v(y - xE[A])}{y(x + v)} < 1 \right\}$$

for policy I and

$$\Omega = \left\{ a \in A, x \in X, y \in Y, v \in V/0 < \frac{v(x + v)(y - xE[A])}{y[x + (x + v)] + v^2} < 1 \right\}$$

for policy II.

Assume that the system characteristic of interest is L_q, the expected number of customers in queue. It follows from (5) the expected number of customers in queue under policy I is

$$f(a, x, y, v) = x \left[\frac{xE[A]}{y(y - xE[A])} + \frac{E[A](A - 1)}{2(yE[A] - xE^2[A])} + \frac{1}{v} \right] \tag{15}$$

The membership function for the expected number of customers in queue under policy I is

$$\eta_{\tilde{L}_q}(z) = \sup_{\Omega} \min \left\{ \frac{\eta_{\tilde{w}}(a), \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{\theta}}(v)}{z = x \left[\frac{xE[A]}{y(y-xE[A])} + \frac{xE[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v} \right]} \right\}. \tag{16a}$$

Likewise, the membership functions for the expected waiting time in the system and the expected length of time the server is idle and busy are

$$\eta_{\tilde{w}_s}(z) = \sup_{\Omega} \min \left\{ \frac{\eta_{\tilde{w}}(a), \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{\theta}}(v)}{z = \left[\frac{xE[A]^2}{xy(y-xE[A])} + \frac{xE[A(A-1)]}{2x(y-xE[A])} + \frac{E[A]}{v} \right]} \right\}. \tag{16b}$$

$$\eta_{\tilde{E}[I]}(z) = \sup_{\omega} \min \left\{ \frac{\eta_{\tilde{w}}(a), \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{\theta}}(v)}{z = x\frac{1}{x} + \frac{1}{v}} \right\}. \tag{16c}$$

$$\eta_{\tilde{E}[B]}(z) = \sup_{\omega} \min \left\{ \frac{\eta_{\tilde{w}}(a), \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y), \eta_{\tilde{\theta}}(v)}{z = x\frac{(x+v)E[A]}{v(y-xE[A])}} \right\}. \qquad (16d)$$

Membership functions of the system characteristics under policy II can be expressed in a similar manner.

Unfortunately, these membership functions are not expressed in the usual forms, makes it very difficult to imagine their shapes. In this paper we approach the representation problem using a mathematical programming technique. Parametric NLPs are developed to find the α-cuts of $f(\tilde{A}, \tilde{\lambda}, \tilde{\mu}, \tilde{\theta})$ based on the extension principle.

3. The Solution Procedure

To re-express the membership function $\eta_{\tilde{q}}$ (for policy I) in an understandable and usual form, we adopt Zadeh's approach, which relies on α-cuts of $\tilde{L}_q$. Definition for the α-cuts of $\tilde{A}, \tilde{\lambda}, \tilde{\mu}$ and $\tilde{\theta}$ as crisp intervals are as follows:

$$A(\alpha) = [a_\alpha^L, a_\alpha^U] = \lfloor \min_{a \in A}\{a/\eta_{\tilde{A}}(a) \geq \alpha\}, \max_{a \in A}\{a/\eta_{\tilde{A}}(a) \geq \alpha\} \rfloor \qquad (17a)$$

$$\lambda(\alpha) = [x_\alpha^L, x_\alpha^U] = \lfloor \min_{x \in X}\{x/\eta_{\tilde{\lambda}}(x) \geq \alpha\}, \max_{x \in X}\{a/\eta_{\tilde{\lambda}}(x) \geq \alpha\} \rfloor \qquad (17b)$$

$$\mu(\alpha) = [y_\alpha^L, x_\alpha^U] = \lfloor \min_{y \in Y}\{y/\eta_{\tilde{\mu}}(y) \geq \alpha\}, \max_{y \in Y}\{a/\eta_{\tilde{\mu}}(y) \geq \alpha\} \rfloor \qquad (17c)$$

$$\theta(\alpha) = [v_\alpha^L, v_\alpha^U] = \lfloor \min_{v \in V}\{v/\eta_{\tilde{\theta}}(v) \geq \alpha\}, \max_{v \in V}\{a/\eta_{\tilde{\theta}}(v) \geq \alpha\} \rfloor \qquad (17d)$$

The constant batch size, arrival, service and vacation rates are shown as intervals when the membership functions are no less than a given possibility level for α. As a result, the bounds of these intervals can be described as functions ofα and can be obtained as: $A_\alpha^L = \min \eta_{\tilde{A}}^{-1}(\alpha), A_\alpha^U = \max \eta_{\tilde{A}}^{-1}(\alpha)$, $X_\alpha^L = \min \eta_{\tilde{\lambda}}^{-1}(\alpha), X_\alpha^U = \max \eta_{\tilde{\lambda}}^{-1}(\alpha)$, $Y_\alpha^L = \min \eta_{\tilde{\mu}}^{-1}(\alpha), Y_\alpha^U = \max \eta_{\tilde{\mu}}^{-1}(\alpha)$, $V_\alpha^L = \min \eta_{\tilde{\theta}}^{-1}(\alpha), V_\alpha^U = \max \eta_{\tilde{\theta}}^{-1}(\alpha)$. Therefore, we can use the α-cuts of $\tilde{L}_q$ to construct its membership function, since the membership function defined in (16a) is parameterized by α.

Using Zadeh's extension principle $\eta_{\tilde{L}_q}(z)$ is the minimum of $\eta_{\tilde{A}}(a), \eta_{\tilde{\lambda}}(x), \eta_{\tilde{\mu}}(y)$ and $\eta_{\tilde{\theta}}(v)$. To derive the membership function $\eta_{\tilde{L}_q}(z)$, we need atleast one of the following cases to hold such that

$$z = x\left[\frac{xE[A]}{y(y - xE[A])} + \frac{E[A(A-1)]}{2(yE[A] - xE^2[A])} + \frac{1}{v}\right]$$

satisfies $\eta_{\tilde{L}_q}(z) = \alpha$.
Case (i) : $(\eta_{\tilde{A}}(x) = \alpha, \eta_{\tilde{\lambda}}(x) \geq \alpha, \eta_{\tilde{\mu}}(y) \geq \alpha, \eta_{\tilde{\theta}}(v) \geq \alpha)$
Case (ii) : $(\eta_{\tilde{A}}(x) \geq \alpha, \eta_{\tilde{\lambda}}(x) = \alpha, \eta_{\tilde{\mu}}(y) \geq \alpha, \eta_{\tilde{\theta}}(v) \geq \alpha)$
Case (iii) : $(\eta_{\tilde{A}}(x) \geq \alpha, \eta_{\tilde{\lambda}}(x) \geq \alpha, \eta_{\tilde{\mu}}(y) = \alpha, \eta_{\tilde{\theta}}(v) \geq \alpha)$
Case (iv) : $(\eta_{\tilde{A}}(x) \geq \alpha, \eta_{\tilde{\lambda}}(x) \geq \alpha, \eta_{\tilde{\mu}}(y) \geq \alpha, \eta_{\tilde{\theta}}(v) = \alpha)$.
This can be accomplished using parametric NLP techniques. The NLP to find the lower and upper bounds of the α-cut of $\eta_{\tilde{L}_q}(z)$ for case (i) are

$$(L_q)_\alpha^{L_1} = \min_\Omega \left[x\left(\frac{xE[A]}{y(y-xE[A])} + \frac{E[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v} \right) \right] \tag{18a}$$

$$(L_q)_\alpha^{U_1} = \max_\Omega \left[x\left(\frac{xE[A]}{y(y-xE[A])} + \frac{E[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v} \right) \right] \tag{18b}$$

For case (ii) are

$$(L_q)_\alpha^{L_2} = \min_\Omega \left[x\left(\frac{xE[A]}{y(y-xE[A])} + \frac{E[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v} \right) \right] \tag{18c}$$

$$(L_q)_\alpha^{U_2} = \max_\Omega \left[x\left(\frac{xE[A]}{y(y-xE[A])} + \frac{E[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v} \right) \right] \tag{18d}$$

For case (iii) are

$$(L_q)_\alpha^{L_3} = \min_\Omega \left[x\left(\frac{xE[A]}{y(y-xE[A])} + \frac{E[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v} \right) \right] \tag{18e}$$

$$(L_q)_\alpha^{U_3} = \min_\Omega \left[x\left(\frac{xE[A]}{y(y-xE[A])} + \frac{E[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v} \right) \right] \tag{18f}$$

and for case (iv) are

$$(L_q)_\alpha^{L_4} = \min_\Omega \left[x\left(\frac{xE[A]}{y(y-xE[A])} + \frac{E[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v} \right) \right] \tag{18g}$$

$$(L_q)_\alpha^{U_4} = \min_\Omega \left[x\left(\frac{xE[A]}{y(y-xE[A])} + \frac{E[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v} \right) \right] \tag{18h}$$

From the definitions of $A(\alpha), \lambda(\alpha), \mu(\alpha)$ and $\theta(\alpha)$ in (17), $a \in \lfloor a_\alpha^L, a_\alpha^U \rfloor, x \in \lfloor x_a^L, x_a^U \rfloor$, $y \in \lfloor y_\alpha^L, y_\alpha^U \rfloor$ and $v \in \lfloor v_\alpha^L, v_\alpha^U \rfloor$ respectively. The α-cuts form a nested structure with respect to (see [19,20]), i.e., given $0 < \alpha_2 < \alpha_1 \leq 1$ we have $\lfloor a_{\alpha_1}^L, a_{\alpha_1}^U \rfloor \subseteq \lfloor a_{\alpha_2}^L, a_{\alpha_2}^U \rfloor$ and $\lfloor x_{\alpha_1}^L, x_{\alpha_1}^U \rfloor \subseteq \lfloor x_{\alpha_2}^L, x_{\alpha_2}^U \rfloor, \lfloor y_{\alpha_1}^L, y_{\alpha_1}^U \rfloor \subseteq \lfloor y_{\alpha_2}^L, y_{\alpha_2}^U \rfloor$ and $\lfloor v_{\alpha_1}^L, v_{\alpha_1}^U \rfloor \subseteq \lfloor v_{\alpha_2}^L, x_{\alpha_2}^U \rfloor$. Therefore, (18a), (18c), (18e) and (18g) have the same smallest element and (18b), (18d), (18f) and

(18h) have the same largest element. To find the membership function $\eta_{\tilde{L}_q}$, it suffices to find the left and right shape functions of $\eta_{\tilde{L}_q}$, which is equivalent of finding the lower bound $(L_q)^L_\alpha$ and upper bound $(L_q)^U_\alpha$ of the α-cuts of $\tilde{L}_q$, which based on (18a), can be written as

$$(L_q)^L_\alpha = \min_{\Omega} x\left[\frac{xE[A]}{y(y-xE[A])} + \frac{E[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v}\right]$$

so that

$$a^L_\alpha \le a \le a^U_\alpha, x^L_\alpha \le x \le x^U_\alpha, y^L_\alpha \le y \le y^U_\alpha, v^L_\alpha \le v \le v^U_\alpha \qquad (19a)$$

$$(L_q)^U_\alpha = \max_{\Omega} x\left[\frac{xE[A]}{y(y-xE[A])} + \frac{E[A(A-1)]}{2(yE[A]-xE^2[A])} + \frac{1}{v}\right]$$

so that

$$a^L_\alpha \le a \le a^U_\alpha, x^L_\alpha \le x \le x^U_\alpha, y^L_\alpha \le y \le y^U_\alpha, v^L_\alpha \le v \le v^U_\alpha \qquad (19b)$$

Atleast one of a, x, y or v must hit the boundaries of their α-cuts to satisfy $\eta_{\tilde{L}_q}(z) = a$. This model is a set of mathematical programs with boundary constraints and tends itself to the systematic study of how the optimal solutions change with $a^L_\alpha, a^U_\alpha, x^L_\alpha, x^U_\alpha, y^L_\alpha, y^U_\alpha, v^L_\alpha$ and v^U_a as varies over (0,1]. This model is a special case of parametric NLPs (see [18]). The crisp interval $\lfloor (L_q)^L_\alpha, (L_q)^U_\alpha \rfloor$ obtained from (19) represents the α-cuts of $\tilde{L}_q$. Again, by applying the results of Zimmermann [19] and Kanfmann [20] and convexity properties to $\tilde{L}_q$, we have $(L_q)^L_{\alpha_1} \ge (L_q)^L_{\alpha_2}$ and $(L_q)^U_{\alpha_1} \le (L_q)^U_{\alpha_2}$. where $0 < \alpha_2 < \alpha_1 < 1$. In other words, $(L_q)^L_\alpha$ increases and $(L_q)^U_\alpha$ decreases as α increases, consequently, the membership function $\eta_{\tilde{L}_q}(z)$ can be found from (19).

If both $(L_q)^L_\alpha$ and $(L_q)^U_\alpha$ are invertible with respect to α, then a left shape function $L(z) = [(L_q)^L_\alpha]^{-1}$ and a right shape function $R(z) = [(L_q)^U_\alpha]^{-1}$ can be derived, from which the membership function $\eta_{\tilde{L}_q}$ is constructed.

$$\eta_{\tilde{L}_q}(z) = \begin{cases} L(z), & (L_q)^L_{\alpha=0} \le z \le (L_q)^L_{\alpha=1}, \\ 1, & (L_q)^L_{\alpha=0} \le z \le (L_q)^U_{\alpha=1}, \\ R(z), & (L_q)^U_{\alpha=0} \le z \le (L_q)^U_{\alpha=0}, \end{cases}$$

In most cases, the values of $(L_q)^L_\alpha$ and $(L_q)^U_\alpha$ cannot be solved analytically. Consequently, a closed -form membership function of $\tilde{L}_q$ cannot be obtained. However, the numerical solutions for $(L_q)^L_\alpha$ and $(L_q)^U_\alpha$ at different possibility levels can be collected to approximate the shapes of $L(z)$ and $R(z)$. That is the set of intervals $\{\lfloor (L_q)^L_\alpha, (L_q)^U_\alpha \rfloor / \alpha \in$

$[0, 1]\}$ shows the shape of $\eta_{\tilde{L}_q}$, although the exact function is not known explicitly. Note that the membership functions for other system characteristics under Policy I, such as be expected waiting time in the system, the expected length of time the server is idle and busy, and the system characteristics under Policy II system can also be derived in a similar manner.

4. Numerical Examples

Example I (The system under Policy I) : Consider a production time in which the production does not start until atleast some specified number of units, $N(N \geq 1)$, are accumulated during an idle period. The number of units arrive in batches. [Using α-cuts the arrival size is a trapezoidal fuzzy number [3 6 9 12] and be interval of confidence be represented by $[3 + 3\alpha, 12 - 9\alpha]$. Using (2) and (3) it is easy to find $E[K]$ and $E[K^2]$]. The operator will take a sequence vacations whenever there are no units to process. The operator of the machine may utilize his vacation time to perform some extra operations such as preventive maintenance or some other work, etc. When the operator returns from a vacation and finds that the number of processed units is less than N, he will go on a vacation again. This worker can be viewed as a server and he or she can take consecutive vacations when there are no units in the system. Concerned with system efficiency, management wants to know the system characteristics, including the number of units in queue, waiting time in the system and the expected length of time the server is idle and busy.

4.1 The Fuzzy Expected Length of Time the Server is Idle

Suppose the batch size, arrival service and vacation rates are trapezoidal fuzzy numbers represented by $\tilde{A} = [3\ \ 6\ \ 9\ \ 12]$, $\tilde{\lambda} = [1\ \ 2\ \ 3\ \ 4]$, $\tilde{\mu} = [3\ \ 4\ \ 5\ \ 6]$ and $\tilde{\theta} = [2\ \ 3\ \ 4\ \ 5]$. First it is easy to find $\lfloor A^L_\alpha, A^U_\alpha \rfloor = [3 + 3\alpha, 12 - 9\alpha]$, $\lfloor x^L_\alpha, x^U_\alpha \rfloor = [1 + \alpha, 4 - \alpha]$, $\lfloor y^L_\alpha, y, U_\alpha \rfloor = [2 + \alpha, 5 - \alpha]$ and $\lfloor v^L_\alpha, v^U_\alpha \rfloor = [.05 + .05\alpha, 5 - .3\alpha]$. Next it is obvious that when $x = x^U_\alpha$ and $v = v^U_\alpha$, the expected length of time the server is idle attains its minimum value and when $x = x^L_\alpha$ and $v = v^L_\alpha$, it attains its maximum value.

The α-cuts of $E[I]$

$$(E(I))^L_\alpha = \frac{9 - 2\alpha}{\alpha^2 - 9\alpha + 20}$$

$$(E[I])^U_\alpha = \frac{3 + 2\alpha}{\alpha^2 + 3\alpha + 2}.$$

The membership function is

$$\eta_{E[I]}(z) = \begin{cases} \frac{(9z-2)\pm\sqrt{z^2+4}}{2z}, & \frac{36}{80} \leq z \leq \frac{28}{48} \\ 1, & \frac{28}{48} \leq z \leq \frac{20}{24} \\ \frac{-(3z-2)\pm\sqrt{z^2+4}}{2z}, & \frac{20}{24} \leq z \leq \frac{12}{8} \end{cases}$$

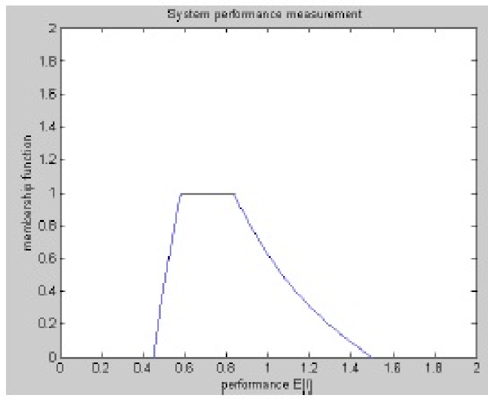

Fig. 1. The membership function for fuzzy expected length of time the server is idle in Example 1.

4.2. The Fuzzy Expected Number of Units in Queue

The batch size arrival rate, service rate and vacation rates are trapezoidal fuzzy numbers represented by $\tilde{A} = [2\ \ 3\ \ 4\ \ 5], \tilde{\lambda} = [1\ \ 2\ \ 3\ \ 4]$, $\tilde{\mu} = [3\ \ 6\ \ 9\ \ 12], \tilde{\theta} = [.05\ \ .1\ \ .2\ \ .5]$. Owing to complicated form of expressions, $(L_q)_\alpha^L$ and $(L_q)_\alpha^U$ cannot be solved analytically. Consequently, a closed-form membership function for $\tilde{L}_q$ cannot be obtained. Hence the software MATLAB 7.0.4 for windows is used for alleviating computational burden. Next we perform α-cuts of batch size, arrival, service and vacation rates and fuzzy expected number of units in queue at eleven distinct α values: $0.1, 0.1, 0.2, \cdots, 1.0$. Crisp intervals for fuzzy expected number of units in queue are presented in table 2. The fuzzy expected number of units in queue has two characteristics to be noted. First the support of $\tilde{L}_q$ ranges from 2.1940 to 77.8095; this indicates that, though the expected number of units in the queue is fuzzy, it is impossible for its value to fall below 2.1940 or exceed 77.8095. Second, the α-cut at $\alpha = 1$ contains the values from 12.0397 to 27.9921, which are the most possible values for the expected number of units in queue.

Table 1 : α-cuts of batch size, arrival, service and vacation rates and fuzzy expected of number of units idle

α	A_α^L	A_α^U	X_α^L	X_α^U	Y_α^L	Y_α^U	V_α^L	V_α^U	$(E[I])_\alpha^L$	$(E[I])_\alpha^U$
0.0000	3.0000	12.0000	1.0000	4.0000	3.0000	6.0000	2.0000	5.0000	0.4500	1.5000
0.1000	3.3000	11.7000	1.1000	3.9000	3.1000	5.9000	2.1000	4.9000	0.4605	1.3853
0.2000	3.6000	11.4000	1.2000	3.8000	3.2000	5.8000	2.2000	4.8000	0.4715	1.2879
0.3000	3.9000	11.1000	1.3000	3.7000	3.3000	5.7000	2.3000	4.7000	0.4830	1.2040
0.4000	4.2000	10.8000	1.4000	3.6000	3.4000	5.6000	2.4000	4.6000	0.4952	1.1310
0.5000	4.5000	10.5000	1.5000	3.5000	3.5000	5.5000	2.5000	4.5000	0.5079	1.0667
0.6000	4.8000	10.2000	1.6000	3.4000	3.6000	5.4000	2.6000	4.4000	0.5214	1.0096
0.7000	5.1000	9.9000	1.7000	3.3000	3.7000	5.3000	2.7000	4.3000	0.5356	0.9586
0.8000	5.4000	9.6000	1.8000	3.2000	3.8000	5.2000	2.8000	4.2000	0.5506	0.9127
0.9000	5.7000	9.3000	1.9000	3.1000	3.9000	5.1000	2.9000	4.1000	0.5665	0.8711
1.0000	6.000	9.0000	2.0000	3.0000	4.0000	5.0000	3.0000	4.0000	0.5833	0.8333

Table 2 : α -cuts of batch size, arrival, service and vacation rates and fuzzy expected number of units in queue

α	A_α^L	A_α^U	X_α^L	X_α^U	Y_α^L	Y_α^U	V_α^L	V_α^U	$(L_q)_\alpha^L$	$(L_q)_\alpha^U$
0.0000	2.0000	5.0000	1.0000	4.0000	3.0000	12.0000	.0500	.5000	2.1940	77.8095
0.1000	2.1000	4.9000	1.1000	3.9000	3.3000	11.7000	.0550	.4700	2.5748	68.8443
0.2000	2.2000	4.8000	1.2000	3.8000	3.6000	11.4000	.0600	.4400	3.0104	61.3648
0.3000	2.3000	4.7000	1.3000	3.7000	3.9000	11.1000	.0650	.4100	3.5138	55.0264
0.4000	2.4000	4.6000	1.4000	3.6000	4.2000	10.8000	.0700	.3800	4.1022	49.5824
0.5000	2.5000	4.5000	1.5000	3.5000	4.5000	10.5000	.0750	.3500	4.7993	44.8513
0.6000	2.6000	4.4000	1.0000	3.4000	4.8000	10.2000	.0800	.3200	5.6342	40.6957
0.7000	2.7000	4.3000	1.7000	3.3000	5.1000	9.9000	.0850	.2900	6.6718	37.0097
0.8000	2.8000	4.2000	1.8000	3.2000	5.4000	9.6000	.0900	.2600	7.9756	33.7087
0.9000	2.9000	4.1000	1.9000	3.1000	5.7000	9.3000	.9500	.2300	9.6821	30.7230
1.0000	3.0000	4.0000	2.0000	3.0000	6.0000	9.0000	.1000	.2000	12.0397	27.9921

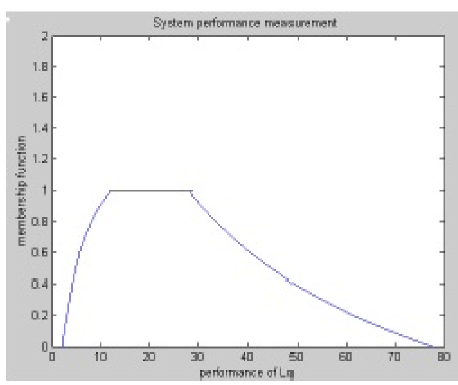

Fig. 2.The membership function for fuzzy expected number of units in queue

4.3 The Fuzzy Expected Waiting Time of Units in the System (ws)

Table 3 : α-cuts of batch size, arrival, service and vacation rates and fuzzy expected waiting time in the system

α	A_α^L	A_α^U	X_α^L	X_α^U	Y_α^L	Y_α^U	V_α^L	V_α^U	$(w_s)_\alpha^L$	$(w_s)_\alpha^U$
0.0000	2.0000	5.0000	1.0000	4.0000	3.0000	12.0000	.0500	.5000	7.9706	69.2500
0.1000	2.1000	4.9000	1.1000	3.9000	3.3000	11.7000	.0550	.4700	8.4916	62.8439
0.2000	2.2000	4.8000	1.2000	3.8000	3.6000	11.4000	.0600	.4400	9.0874	57.4924
0.3000	2.3000	4.7000	1.3000	3.7000	3.9000	11.1000	.0650	.4100	9.7755	52.9495
0.4000	2.4000	4.6000	1.4000	3.6000	4.2000	10.8000	.0700	.3800	10.5795	49.0385
0.5000	2.5000	4.5000	1.5000	3.5000	4.5000	10.5000	.0750	.3500	11.5317	45.6290
0.6000	2.6000	4.4000	1.0000	3.4000	4.8000	10.2000	.0800	.3200	12.6788	42.6218
0.7000	2.7000	4.3000	1.7000	3.3000	5.1000	9.9000	.0850	.2900	14.0896	39.9390
0.8000	2.8000	4.2000	1.8000	3.2000	5.4000	9.6000	.0900	.2600	15.8726	37.5170
0.9000	2.9000	4.1000	1.9000	3.1000	5.7000	9.3000	.9500	.2300	18.2117	35.3013
1.0000	3.0000	4.0000	2.0000	3.0000	6.0000	9.0000	.1000	.2000	21.4583	33.2407

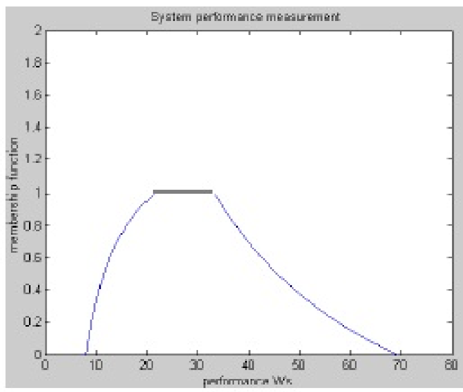

Fig. 3. The membership function of fuzzy expected waiting time of units in the system

4.4. The Fuzzy Expected Length of Time the Server is Busy $(E[B])$

Table 4 : α-cuts of batch size, arrival, service and vacation rates and fuzzy expected length of time the server is busy

α	A_α^L	A_α^U	X_α^L	X_α^U	Y_α^L	Y_α^U	V_α^L	V_α^U	$(E[B])_\alpha^L$	$(E[B])_\alpha^U$
0.0000	1.0000	2.500	3.000	6.000	19.000	22.000	0.0500	0.5000	0.7313	24.9118
0.1000	1.0500	2.450	3.100	5.900	19.100	21.900	0.0550	0.4700	0.8068	21.5929
0.2000	1.1000	2.400	3.200	5.800	19.200	21.800	0.0600	0.4400	0.8937	18.8858
0.3000	1.1500	2.350	3.300	5.700	12.300	21.700	0.0650	0.4100	0.9944	16.6447
0.4000	1.2000	2.300	3.400	5.000	19.400	21.600	0.0700	0.3800	1.1123	14.7656
0.5000	1.2500	2.250	3.500	5.500	19.500	21.500	0.0750	0.3500	1.2520	13.1730
0.6000	1.3000	2.200	3.600	5.400	19.600	21.400	0.0800	0.3200	1.4197	11.8103
0.7000	1.3500	2.150	3.700	5.300	19.700	21.300	0.0850	0.2900	1.6241	10.6348
0.8000	1.4000	2.1000	3.800	5.200	19.800	21.200	0.0900	0.2600	1.8781	9.6132
0.9000	1.4500	2.0500	3.900	5.100	19.900	21.100	0.0950	0.2300	2.2013	8.7196
1.0000	1.5000	2.0000	4.000	5.000	20.000	21.000	0.1000	0.2000	2.6250	7.9333

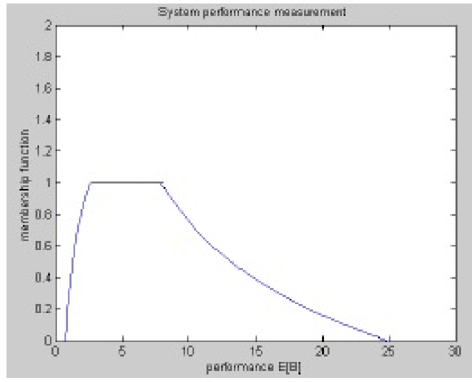

Fig. 4. The membership function of the expected length of time the server is busy

Example 2 (The system under Policy II):

Table 5 : α-cuts of batch size arrival, service and vacation rates and expected length of units in queue

α	A_α^L	A_α^U	X_α^L	X_α^U	Y_α^L	Y_α^U	V_α^L	V_α^U	$(L_q)_\alpha^L$	$(L_q)_\alpha^U$
0.0000	0.0500	0.5000	1.0000	2.5000	3.0000	6.0000	19.0000	22.0000	1.9083	77.7972
0.1000	0.0550	0.4700	1.0500	2.4500	3.1000	5.9000	19.1000	21.9000	2.3094	68.8304
0.2000	0.0600	0.4400	1.1000	2.4000	3.2000	5.8000	19.2000	21.8000	2.7662	61.3493
0.3000	0.0650	0.4100	1.1500	2.3500	3.3000	5.7000	12.3000	21.7000	3.2909	55.0092
0.4000	0.0700	0.3800	1.2000	2.3000	3.4000	5.0000	19.4000	21.6000	3.9004	49.5634
0.5000	0.0750	0.3500	1.2500	2.2500	3.5000	5.5000	19.5000	21.5000	4.6181	44.8303
0.6000	0.0800	0.3200	1.3000	2.2000	3.6000	5.4000	19.6000	21.4000	5.4779	40.6728
0.7000	0.0850	0.2900	1.3500	2.1500	3.7000	5.3000	19.7000	21.3000	6.5296	36.9846
0.8000	0.0900	0.2600	1.4000	2.1000	3.8000	5.2000	19.8000	21.2000	7.8516	33.6814
0.9000	0.0950	0.2300	1.4500	2.0500	3.9000	5.1000	19.9000	21.1000	9.5755	30.6933
1.0000	0.1000	0.2000	1.5000	2.0000	4.0000	5.0000	20.0000	21.0000	11.9496	27.9598

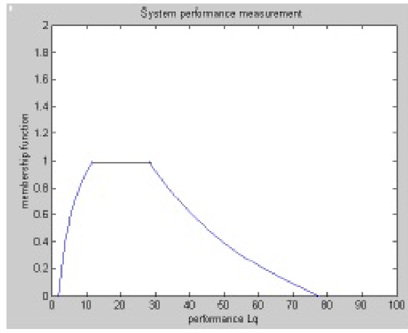

Fig. 5.The membership function of expected length of queue

Table 6 : α-cuts of batch size, arrival, service and vacation rates and expected waiting time of units in the system

α	A^L_α	A^U_α	X^L_α	X^U_α	Y^L_α	Y^U_α	V^L_α	V^U_α	$(w_s)^L_\alpha$	$(w_s)^U_\alpha$
0.0000	0.0500	0.5000	1.0000	2.5000	3.0000	6.0000	19.0000	22.0000	0.0137	0.4098
0.1000	0.0550	0.4700	1.0500	2.4500	3.1000	5.9000	19.1000	21.9000	0.0197	0.2356
0.2000	0.0600	0.4400	1.1000	2.4000	3.2000	5.8000	19.2000	21.8000	0.0259	0.1925
0.3000	0.0650	0.4100	1.1500	2.3500	3.3000	5.7000	12.3000	21.7000	0.0323	0.1723
0.4000	0.0700	0.3800	1.2000	2.3000	3.4000	5.0000	19.4000	21.6000	0.0387	0.1602
0.5000	0.0750	0.3500	1.2500	2.2500	3.5000	5.5000	19.5000	21.5000	0.0452	0.1518
0.6000	0.0800	0.3200	1.3000	2.2000	3.6000	5.4000	19.6000	21.4000	0.0516	0.1452
0.7000	0.0850	0.2900	1.3500	2.1500	3.7000	5.3000	19.7000	21.3000	0.0580	0.1397
0.8000	0.0900	0.2600	1.4000	2.1000	3.8000	5.2000	19.8000	21.2000	0.0642	0.1348
0.9000	0.0950	0.2300	1.4500	2.0500	3.9000	5.1000	19.9000	21.1000	0.0703	0.1303
1.0000	0.1000	0.2000	1.5000	2.0000	4.0000	5.0000	20.0000	21.0000	0.0763	0.1259

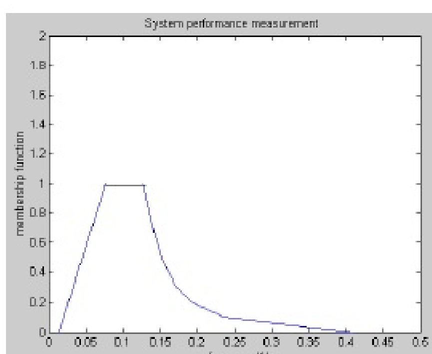

Fig. 6.The membership function of expected waiting time in the system

Table 7 : α-cuts of batch size, arrival, service and vacation rates and expected length of time the server is busy

α	A_α^L	A_α^U	X_α^L	X_α^U	Y_α^L	Y_α^U	V_α^L	V_α^U	$(E[B])_\alpha^L$	$(E[B])_\alpha^U$
0.0000	1.0000	2.5000	3.0000	6.0000	19.0000	22.0000	0.0500	0.5000	0.6418	24.7076
0.1000	1.0500	2.4500	3.1000	5.9000	19.1000	21.9000	0.0550	0.4700	0.7146	21.3953
0.2000	1.1000	2.4000	3.2000	5.8000	19.2000	21.8000	0.0600	0.4400	0.7987	18.6944
0.3000	1.1500	2.3500	3.3000	5.7000	19.3000	21.7000	0.0650	0.4100	0.8966	16.4591
0.4000	1.2000	2.3000	3.4000	5.6000	19.4000	21.6000	0.0700	0.3800	1.0117	14.5856
0.5000	1.2500	2.2500	3.5000	5.5000	19.5000	21.5000	0.0750	0.3500	1.1486	12.9982
0.6000	1.3000	2.2000	3.6000	5.4000	19.6000	21.4000	0.0800	0.3200	1.3133	11.6404
0.7000	1.3500	2.1500	3.7000	5.3000	19.7000	21.3000	0.0850	0.2900	1.5147	10.4696
0.8000	1.4000	2.1000	3.8000	5.2000	19.8000	21.2000	0.0900	0.2600	1.7656	9.4524
0.9000	1.4500	2.0500	3.9000	5.1000	19.9000	21.1000	0.0950	0.2300	2.0856	8.5630
1.0000	1.5000	2.0000	4.0000	5.0000	20.0000	21.0000	0.1000	0.2000	2.5060	7.7808

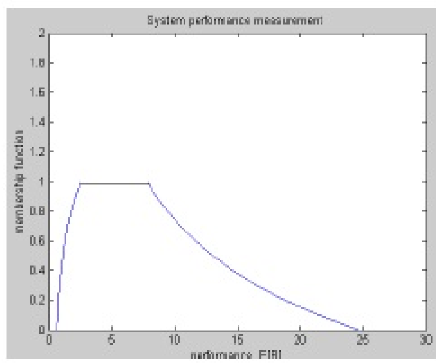

Fig. 7. The membership function of fuzzy expected time the server is busy

Table 8 : α-cuts of batch size, arrival, service and vacation rates and expected length of time the server is idle

α	A_α^L	A_α^U	X_α^L	X_α^U	Y_α^L	Y_α^U	$(E[I])_\alpha^L$	$(E[I])_\alpha^U$
0.0000	3.0000	12.0000	1.0000	4.0000	3.0000	6.0000	0.3389	1.1667
0.1000	3.3000	11.7000	1.1000	3.9000	3.1000	5.9000	0.3469	1.0728
0.2000	3.6000	11.4000	1.2000	3.8000	3.2000	5.8000	0.3552	0.9938
0.3000	3.9000	11.1000	1.3000	3.7000	3.3000	5.7000	0.3640	0.9262
0.4000	4.2000	10.8000	1.4000	3.6000	3.4000	5.6000	0.3742	0.8678
0.5000	4.5000	10.5000	1.5000	3.5000	3.5000	5.5000	0.3829	0.8167
0.6000	4.8000	10.2000	1.6000	3.4000	3.6000	5.4000	0.3932	0.7715
0.7000	5.1000	9.9000	1.7000	3.3000	3.7000	5.3000	0.4040	0.7313
0.8000	5.4000	9.6000	1.8000	3.2000	3.8000	5.2000	0.4155	0.6953
0.9000	5.7000	9.3000	1.9000	3.1000	3.9000	5.1000	0.4276	0.6628
1.0000	6.0000	9.0000	2.0000	3.0000	4.0000	5.0000	0.4405	0.6333

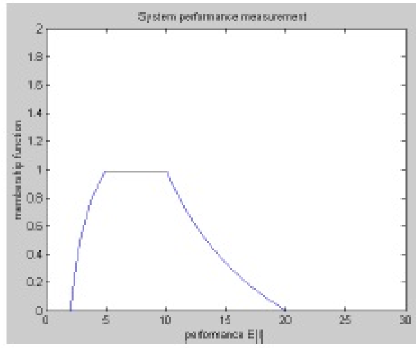

Fig. 8. The membership function of fuzzy expected time the server is idle

The results of Examples 1 and 2 shows that the range of the membership functions under Policy I and Policy II.

5. Conclusion

This paper applies the concept of α-cut approach and Zadeh's extension principle to construct the membership function of expected number of units in queue, waiting time in the system, expected length of time the server is idle and expected length of time the server is busy using paired NLP methods. By using α-cut approach, the membership functions are found and their interval limits are inverted to attain explicit closed-form expressions for the system characteristics. Even when the membership function intervals cannot be inverted explicitly, system manager (or) designers can also specify the system characteristics of interest, perform numerical results to examine the corresponding α-cuts and then use this information to develop (or) improve system processes.

For example, in example 1, a manager can set the range of waiting time of units in the system [10.5795, 49.0385] to reflect be desired batch size, arrival rate and service rate and find that the corresponding α-level is 0.4000 with $A_\alpha^L = 2.4000, A_\alpha^U = 4.6000, X_\alpha^L = 1.4000, X_\alpha^U = 3.6000, Y_\alpha^L = 4.2000, Y_\alpha^U = 10.8000$. In other words the manager can determine the batch size is between 2.4000 and 4.7000, the arrival rate is between 1.40000 and 3.0000 and the service rate is between 4.2000 and 10.8000. As this example demonstrates, the approach proposed in this paper provides practical information for system designers and practitioners.

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