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ON SEMI UNIFORM SPACE

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Abstract
In this paper, we introduce the notions of uniform space and study these notions on a semi uniform space. Also We stated and prove some theorems which determine the relationship between these notions and some types of semi uniform space.

Introduction

Before Andre Weil gave the first explicit definition of a uniform structure in 1937, uniform concepts, like completeness, were discussed using metric spaces. Nicolas Bourbaki [4] provided the definition of uniform structure in terms of entourages in the book Topologie Generale1955 [3] and John Tukey gave the uniform cover definition. Weil also characterized uniform spaces in terms of a family of pseudo metrics. In this paper will use the entourage or Bourbaki approach to uniform spaces as presented in Kelley [3]. The equivalent Tukey approach via uniform coverings, which some find more intuitive, can be found in Isbell [1]. The paper is organized as follows: In Section 1 we introduce

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Key Words : *Uniform space, Semi uniform space,* u_1 *- semi uniform space,* u_2 *-semi uniform space.*

some definitions and results on uniform space which we use them in this work. In Section 2 we introduce definition of semi uniform space, u_1- semi uniform space, u_2-semi uniform space and the relationship between them.

1. Preliminaries

The aim of this section gives some definitions and results on uniform space which we use them in this work.

Definition 1.1 [1] : Let X be a non-empty set and let $U \subseteq X \times X$ and $V \subseteq X \times X$, we write

(1) $U^{-1} = \{(y, x)/(x, y) \in U\}$

(2) $U \circ V = \{(x, y) \in X \times X\}/(x, z) \in U, (z, y) \in V$

(3) $\Delta = \{(x, x)/x \in X\}$ it is called the diagonal of X.

Remark 1.2 : Let X be a non-empty set and let $U \subseteq X \times X$.
If $\Delta \subseteq U$ then $U \subseteq U \circ U$.

Proof : Let $(x, y) \in U$.
Since $\Delta \subseteq U$ then $(x, x) \in U$ and since $(x, y) \in U$ then $(x, y) \in U = circU$.
Then $U \subseteq U \circ U$.

Definition 1.3 [3] : Let X be a non-empty set. A uniformity for X is u, a non-empty set of relations on X, such that

(i) For all $U \in u, \Delta \subseteq U$.

(ii) $U \in u$ and $U \subseteq W \subseteq X \times X \Rightarrow W \in u$

(iii) $U \in u$ and $V \in u \Rightarrow U \cap V \in u$

(iv) $U \in u \Rightarrow U^{-1} \in u$

(v) $U \in u \Rightarrow \exists V \in u$ with $V \circ V \subseteq U$

the pair (X, u) is called uniform space and the element of u is called encourage.

Definition 1.4 [2] : Let X be a non-empty set. A uniformity sub base for X is β, a non-empty set of relations on X, such that

(i) $\forall U \in \beta, \Delta \subseteq U$

(ii) $U \in \beta \Rightarrow U^{-1} \in \beta$

(iii) $U \in \beta \Rightarrow \exists V \in \beta$ with $V \circ V \in U$.

Definition 1.5 [2] : The uniformity sub base β is called uniformity base iff $U \cap V \in \beta \ \forall \ U, V \in \beta$.

Examples 1.6 [1] : Let X be non-empty set

(1) $u = \{X \times X\}$ is called indiscrete uniformity.

(2) $u = \{U \subseteq X \times X / \Delta \subseteq U\}$ is called discrete uniformity.

(3) Let (X, d) be metric space. Then the set $\beta = \{U_\epsilon, \epsilon > 0\}$ is uniformity sub base for X where $U_\epsilon = \{(x, y)/d(x, y) < \epsilon\}$.

Definition 1.7 : Let X be a non-empty set and let $\Delta \subseteq S \subseteq X \times X$ s.t. $S \circ S$ and $S^{-1} = S$ we define the uniformity $u = \{U \subseteq X \times X/S \subseteq U\}$ is called and the pair (X, u) is called S-uniform space.

Examples 1.8 :

(1) The discrete uniform space and indiscrete uniform space are S-uniform space.

(2) Let $X = \{1, 2, 3\}$ and let $S = \Delta \cup \{(1, 2), (2, 1)\}$ then the uniformity $u = \{U \subseteq X \times X/S \subseteq U\}$ is S-uniformity.

Theorem 1.9 : Let (X, u) be uniform space. If X is finite set then u is S-uniformity.

Proof : Since X is finite set then u is finite collection then $\bigcap\limits_{A \in u} A \in u$.

If $\bigcap\limits_{A \in u} A = \Delta$ then $\Delta \in u$ [since is finite collection] then u is S-uniformity.

If $\bigcap\limits_{A \in u} A \neq \Delta$ then $\exists \ S \neq \Delta \exists \bigcap\limits_{A \in u} A = S$.

Clear $S \subseteq A \ \forall \ A \in u$ then $S^{-1} \subseteq A^{-1} \ \forall \ A \in u$.

Since $\bigcap\limits_{A \in u} A = S \in u$, then $S = S^{-1}$. □

2. The Main Results

In this section, we define the notions of a semi uniform space and we study the relation between semi uniform space and uniform space . We introduce some properties of semi uniform space u_1, u_2. For our discussion, we shall link these notions with other notions which mentioned in preliminaries.

Definition 2.1 : Let X be a non-empty set. A semi uniformity for X is u, a non-empty set of relations on X, such that

(i) $\forall\, U \in u,\ \Delta \subseteq U$

(ii) $U \in u$ and $U \subseteq W \subseteq X \times X \Rightarrow W \in u$

(iii) $U \in u \Rightarrow U^{-1} \in u$

(iv) $U \in u \Rightarrow \exists\, V \in u$ with $V \circ V \subseteq U$.

and the pair (X, u) is called semi uniform space.

Definition 2.2 : A semi uniformity base is the uniformity sub base β.

Example 2.3 : Let $X = \{1, 2, 3\}$, and let

$$\beta = \{X \times X - \{(1,2),(1,3)\}, X \times X - \{(2.1),(3,1)\}$$

then β is semi uniformity base for X.

Remark 2.4 : Clear if X is uniform space then X is semi uniform space and the converse it is not true in general as the following examples.

Example 2.5 : Let $X = \{1, 2, 3\}$ and let

$$\begin{aligned} u \;=\; & \{X \times X, X \times X - \{(1,2)\}, X \times X - \{(2,1)\}, X \times X - \{(1,3)\}, X \times X - \{(3,1)\}, \\ & X \times X - \{(1,2),(1,3)\}, X \times X - \{(2,1),(3,1)\}\} \end{aligned}$$

then u is semi uniformity but not uniformity sice $X \times X - \{1,2),(1,3)\} \in u$ and $X \times X - \{(2,1),(3,1)\} \in u$ but

$$X \times X - \{(1,2),(1,3)\} \cap X \times X - \{(2,1),(3,1)\} = X \times X - \{(1,2),(1,3),(2,1),(3,1)\} \notin u.$$

Example 2.6 : Let $X = R$ and let $u = \{B \subseteq R \times R / \Delta \subset B\}$.
u is semi uniformity but not uniformity since $\Delta \cup \{(1,0)\} \in u$ and $\Delta \cup \{(2,3)\} \in u$, but $\Delta \cup \{(1,0)\} \cap \Delta \cup \{(2,3)\} = \Delta \notin u$ [since $\Delta \not\subset \Delta$].

Proposition 2.7 : Let X be non-empty set / $ord(X) \geq 3$. If $U = X \times X - \{(x,y)\}$ then $U \circ U = X \times X$.

Proof : Let $ord(X) \geq 3$ and let $U = X \times X - \{(x, y)\}$ since $ord(X) \geq 3$ then $\exists\ z \in X$ s.t. $z \neq x$ and $z \neq y$.
Since $U = X \times X - \{(x, y)\}$ then $(x, z) \in U$ and $(z, y) \in U$.
Then $(x, y) \in U \circ U$. So $U \circ U = X \times X$. □

Example 2.8 : Let $X = \{1, 2, 3, 4\}$ and let $U = X \times X - \{(1, 2), (3, 4)\}$ since $\Delta \subseteq U \Rightarrow U \subseteq U \circ U$ by Remark 1.2.
Since $(1, 3)(3, 2) \in U \Rightarrow (1, 2) \in U \circ U$ and since $(3, 2), (2, 4) \in U) \Rightarrow (3, 4) \in U \circ U$ then $U \circ U = X \times X$.
Then in general by Example 2.8 Proposition 2.7 is not necessary true.

Corollary 2.9 : Let (X, u) be semi uniform space / $ord(X) \geq 3. If U = X \times X - \{(x, y)\} \in u$ then $\exists\ V \in u/V \subset U$ and $V \circ V \subseteq U$.

Proof : Let (X, u) be semi uniform space / $ord(X) \geq 3$ and let $U = X \times X - \{(x, y)\} \in u$ then $\exists\ V \in u/V \circ V \subseteq U$ [since $X, u)$ be semi uniform space].
$\because\ \ V \subseteq V \circ V$ [since $\Delta \subseteq V$]
from by Proposition 2.7, we have $U \circ U = X \times X \nsubseteq U$ then $V \subset U$ s.t. $V \circ V \subseteq U$. □

Theorem 2.10 : Let (X, u) be semi uniform space / $ord(X) \geq 3$ and let $A \in u$. If $\Delta \subset A \circ A \subset X \times X$ then $\exists\ B \in u$ s.t. $A \subseteq B$ and $B^{-1} \neq B \subset X \times X$.

Proof : Let (X, u) be semi uniform space / $ord(X) \geq 3$ and let $A \in u$ and let $A \circ A \neq \Delta$ and $A \neq X \times X$ since $A \circ A \neq X \times X$ and $A \subseteq A \circ A$ then $A \neq X \times X$ if $A^{-1} \neq A$ end the proof
If $A^{-1} = A$ since $A \neq X \times X$ then $\exists\ (x, y) \in X \times X/(x, y) \notin A$.
Put $B = A \cup \{(x, y)\}$ then $B \in u$ [since $a \subseteq B$] if $B = X \times X$ then $A = X \times X - \{(x, y)\}$ since $ord(X) \geq 3$ then by Proposition 2.5 we have $A \circ A = X \times X \cdots$ a contradiction
Then $B \subset X \times X$, since $A \subseteq B$ and $B^{-1} = A \cup \{(y, x)\} \neq B$ [since $A = A^{-1}$]. □

Definition 2.11 : Let (X, u) is semi uniform space.
X is u_1 iff for all $(x, y) \neq (z, w)/(x, y) \notin \Delta, (z, w) \notin \Delta\ \exists\ U \in u$ and $V \in u$ such that
$((x, y) \in U$ and $(z, w) \notin U$.
$((z, w) \in V$ and $(x, y) \notin V$.

Definition 2.12 : Let (X, u) is semi uniform space.
X is u_2 iff for all $(x, y) \neq (z, w)/(x, y) \notin \Delta, (x, w) \notin \Delta$.
$\exists\ U \in u$ and $V \in u$ such that $(x, y) \in U$ and $(z, w) \in V/U \cap V = \Delta$.

Examples 2.13 :

(1) The discrete uniformity space is u_1 and u_2.

(2) The indiscrete uniformity space on a set has at least two elements is not u_1 and u_2.

(3) Let $X = R$ and let $u = \{B \subseteq R \times R / \Delta \subset B\}$ since for all $(x, y) \neq (z, w)/(x, y) \notin \Delta, (z, w) \notin \Delta$.

Since $(x, y) \in \Delta \cup \{(x, y)\}$ and $(z, w) \notin \Delta U\{(x, y)\}$, $(z, w) \in \Delta \cup \{(z, w)\}$ and $(x, y) \notin \Delta \cup \{(z, w)\}$ and $\Delta \cup \{(x, y)\} \cap \Delta \cup \{(z, w)\} = \Delta$ then X is u_1 and u_2.

Theorem 2.14 : Let $(X; u)$ is semi uniform space. If X is u_2 then X is u_1.

Proof : Since X is u_2 then for all $(x, y) \neq (z, w)/(x, y), (z, w) \notin \Delta \; \exists \; U \in u$ and $V \in u$ such that $(x, y) \in U$ and $(z, w) \in V/U \cap V = \Delta$.

Since $(x, y) \in U$ and $(z, w) \in V/U \cap V = \Delta$ then $(x, y) \in U$ and $(z, w) \notin U(z, w) \in V$ and $(x, y) \notin V$ then X is u_1.

Example 2.15 : Let $X = \{1, 2, 3\}$ and define the uniformity u

$$\begin{aligned} u = \; & \{X \times X, X \times X - \{(1, 2)\}, X \times X - \{(1, 3)\}, X \times X - \{(2, 1)\}, X \times X - \{(3, 1)\}, \\ & X \times X - \{(2, 3)\}, X \times X - \{(3, 2)\}, X \times X - \{(1, 2), 1, 3)\}, X \times X - \{(2, 1), (3, 1)\}, \\ & X \times X - \{(1, 3), 2, 3)\}, X \times X - \{(3, 1), (3, 2)\}\} \end{aligned}$$

Clear X is u_1.

But not u_2 [since $\forall \; U \in u$ and $V \in u/(1, 2) \in U$ and $(1, 3) \in V, U \cap V \neq \Delta$] then conversely Theorem 2.14 in not true.

Theorem 2.16 : Let (X, u) be S - uniform space. Then X is u_2 iff $S = \Delta$.

Proof : Let (X, u) is S-uniform space.

$(\Rightarrow)$ X is u_2 and let $S \neq \Delta$.

$\because \quad U \cap V \neq \Delta$ [since $S \subseteq U \cap V$ for all $U \in u$ and $V \in u$].

Hence X is not u_2, this complete the proof.

$(\Leftarrow)$ clear by the definition.

Definition 2.17 : Let (X, u) be semi uniform space and let $Y \subseteq X/Y \neq \phi$, we define $u_Y = \{Y \times Y \cap B/B \in u\}$ and we called the pair (Y, u_Y) sub semi uniform space.

Example 2.18 : Let $X = \{1, 2, 3\}$.

$$\begin{aligned} u = \; & \{X \times X, X \times X - \{(1, 2)\}, X \times X - \{(2, 1)\}, X \times X - \{(1, 3)\}, \\ & X \times X - \{(3, 1)\}, X \times X - \{(1, 2), (1, 3)\}, X \times X - \{(2, 1), (3, 1)\}\} \end{aligned}$$

take $Y = \{1, 2\}$ then $u_Y = \{Y \times Y, Y \times Y - \{(1, 2)\}, Y \times Y - \{(2, 1)\}\}$ since

$$Y \times Y \cap X \times X = Y \times Y \cap X \times X - \{(1, 3)\} = Y \times Y \cap X \times X - \{(3, 1)\} = Y \times Y$$

$$Y \times Y \cap X \times X - \{(1, 2)\} = Y \times Y \cap X \times X - \{(1, 2), (1, 3)\} = Y \times Y - \{(1, 2)\}$$

$$Y \times Y \cap X \times X - \{(2, 1)\} = Y \times Y \cap X \times X - \{(2, 1), (3, 1)\} = Y \times Y - \{(2, 1)\}.$$

Theorem 2.19 : Let (X, u) be semi uniform space and let $Y \subseteq X/Y \neq \phi$.

(1) If X is u_1 then Y is u_1.

(2) If X is u_2 then Y is u_2.

Proof : Let (X, u) be semi uniform space and let $Y \subseteq X/Y \neq \phi$

(1) Let X is u_1 and let $(x, y) \neq (z, w)/(x, y), (z, w) \in Y \times Y/\Delta_Y$ [where Δ_Y is the diagonal of Y] since $Y \times Y \subseteq X \times X$ then $(x, y) \in X \times X, (z, w) \in X \times X$.

Since X is u_1 then $\exists\ U \in u$ and $V \in u/(x, y) \in U, (z, w) \notin U$ and $(z, w) \in V, (x, y) \notin V$ then $(x, y) \in Y \times Y \cap U$ and $(z, w) \notin Y \times Y \cap U$ and $(z, w) \in Y \times Y \cap V$ and $(x, y) \notin Y \times Y \cap V$ then Y is u_1.

(2) Let X is u_2 and let $(x, y) \neq (z, w)/(x, y) \in Y \times Y, (z, w) \in Y \times Y$ and $x \neq y, z \neq w$ since $Y \times Y \subseteq X \times X$ then $(x, y) \in X \times X, (z, w) \in X \times X$.

Since X is u_2 then $\exists\ A \in u$ and $B \in u/(x, y) \in A, (z, w) \in B$ and $A \cap B = \Delta_X$ [where Δ_X is the diagonal of X] then $(x, y) \in Y \times Y \cap A$ and $(z, w) \in Y \times Y \cap B$.

Snce $A \cap B = \Delta_X$ then $(Y \times Y \cap A) \cap (Y \times Y \cap B) = \Delta_Y$ then Y is u_2. □

Example 2.20 : Let $X = \{1, 2, 3\}$ and $u = \{A \subseteq X \times X/\Delta \cup \{(1, 2), (2, 1)\} \subset A\}$, then X is not u_1 and not u_2 since $(1, 2), (2, 1) \in U$ for all $U \in u$.
Tae $Y = \{1, 3\}$ then u_Y is discrete uniformity. Clear Y is u_1 and u_2 then the conversely of Theorem 2.19 is not necessary true.

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