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SOME NEW TYPES OF FILTERS IN BCK-ALGEBRA

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Abstract

In this paper, we study a new types of filters its called complete BCK-filter denoted by (c-BCK-filter), complete crazy filter denoted by (c-crazy filter) and closed with respect to c-BCK filter. Also we stated and prove some theorems which determine the relationship between these notions.

1. Introduction

In 1966, Y. Imai and K. Iseki introduced the a new notation called a BCK-algebra [7], thereafter in 1980, E. Y. Deeba [1] introduced the notation of filters and in the setting of bounded implicative BCK-algebra constructed quotient algebra via a filter. In 1996 J. Meng [2] introduced the notion of BCK-filter in BCK-algeba. Then, after in 2007 Wieslaw Dudek and young Bae Jun gave the definition of the poor and crazy filters see [6]. The paper is organized as follows, in section 1 we introduced some definitions and results on BCK-algebra which we use in this paper. In section 2 we introduced definitions of completely BCK filter, completely crazy filter, closed with respect to c-BCK filter and relationship between them.

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2. Preliminaries

In this section, we give some basic concepts about BCK-algebra, BCK-filter, crazy filter and basic concepts that we need in our work.

Definition 2.1 [7] : A BCK-algebra is a set X with a binary operation "$*$" and constant "0" which satisfies the following axioms :

(1) $((x * y) * (x * z)) * (z * y) = 0$,

(2) $(x * (x * y)) * y = 0$,

(3) $x * x = 0$,

(4) $0 * x = 0$,

(5) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Remark 2.2 [7] : A BCK-algebra can be (partially) ordered by $x \leq y$ if and only if $x * y = 0$.

Theorem 2.3 [7] : In any BCK-algebra, the following hold :

(1) $x * 0 = x$,

(2) $x * y \leq x$,

(3) $(x * y) * z = (x * z) * y$,

(4) $(x * z) * (y * z) \leq x * y$,

(5) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,

for all $x, y, z \in X$.

Definition 2.4 [4] : If there is a special element e of a BCK-algebra X satisfying $x \leq e$ for all $x \in X$ then e is called a unit of X. A BCK-algebra with unit is called bounded BCK-algebra. In a bounded BCK-algebra X, we denoted by for every $e * x$ by x^* for every $x \in X$.

Definition 2.5 [2] : A BCK-algebra X is said to be **commutative** if satisfies $x*(x*y) = y * (y * x)$ and for all $x, y \in X, y * (y * x) = x \wedge y$ and $(x^* \wedge y^*)^* = x \vee y$.

Definition 2.6 [**4**] : For a bounded BCK-algebra, if an element x satisfies $(x^*)^* = x$, then x is called an involution . If every element of X is an involution, we call X is an involutory BCK-algebra.

Proposition 2.7 [**3**] : A bounded commutative BCK-algebra is involutory.

Proposition 2.8 [**3**] : In a bounded BCK-algebra, we have

(1) $e^* = 0$ and $0^* = e$,

(2) $y \leq x$ implies $x^* \leq y^*$,

(3) $x^* * y^* \leq y * x$,

(4) $x^* * y^* = y * x$, when X is invoultary (commutative),

Definition 2.9 [**3**] : A nonempty subset F of a bounded BCK-algebra X is called BCK-filter if

(1) $e \in F$

(2) $(x^* * y^*)^* \in F$, $y \in F$ implies $x \in F$.

Definition 2.10 [**5**] : A nonempty subset F of bounded BCK-algebra X is called a crazy filter if it satisfied

(1) $e \in F$,

(2) $(y * x)^* \in F$, $y \in F$ implies $x \in F$.

Definition 2.11 [**4**] : Let f be a mapping from a BCK-algebra X into a BCK-algebra Y. Then f is called

(1) Homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in Y$.

(2) Epimorphism if f is homomorphism and onto.

(3) Monomorphism if f is homomorphism and one to one.

(4) Isomorphism if f is epimorphism and monomorphism.

Lemma 2.12 [**4**] : If f is a homomorphism from BCK-algebra X into BCK-algebra Y, then f is isotone i.e, $x \le y \Rightarrow f(x) \le f(y)$, for all $x, y \in X$.

Lemma 2.13 [**5**] : Suppose f is an epimorphism from BCK-algebra X into BCK-algebra Y, then $f(e_x) = e_y$ where e_x, e_y are the units of X and Y respectively.

Lemma 2.14 : let f be an isomorphism from BCK-algebra X into BCK-algebra Y, then

(1) $f(x^*) = (f(x))^*$, for all $x \in X$.

(2) $f^{-1}(y^*) = (f^{-1}(y))^*$, for all $y \in Y$.

Proof :

(1) $f(x^*) = f(e_x * x) = f(e_x) * f(x) = e_y * f(x) = (f(x))^*$ by Lemma 2.13.

(2) $f^{-1}(y^*) = f^{-1}(e_y * y) = f^{-1}(e_y) * f^{-1}(y) = e_x * f^{-1}(y) = (f^{-1}(y))^*$ by lemma 2.13.

Theorem 2.15 : If f is an isomorphism from BCK-algebra X into BCK-algebra Y, then the image of a crazy filter is a crazy filter.

Proof : Let f be an isomorphism from BCK-algebra X into BCK-algebra Y and let A be a crazy filter in X, then $e_x \in A$ so $f(e_x) = e_y \in f(A)$, (by Lemma 2.13).
Now let $(y * x)^* \in f(A), y \in f(A)$, then $f^{-1}((y * x)^*) \in A, f^{-1}(y) \in A$ (since f is onto).
But $f^{-1}((y * x)^*) = (f^{-1}(y * x))^* = (f^{-1}(y) * f^{-1}(x))^*$ by (Lemma 2.14(2)).
Therefore , $(f^{-1}(y) * f^{-1}(x))^* \in A, f^{-1}(y) \in A$.
Since A is a crazy filter in X, then $f^{-1}(x) \in A$, thus $x \in f(A)$, that means $f(A)$ is a crazy filter in Y.

Theorem 2.16 : If f is an epimorphism from BCK-algebra X into BCK-algebra Y, then the inverse image of a crazy filter is a crazy filter.

Proof : Let f be an epimorphism from BCK-algebra X into BCK-algebra Y and let B be a crazy filter in Y. Then $e_y \in B$ since $f(e_x) = e_y \in B$, (by Lemma 2.13), thus $e_x \in f^{-1}(B)$.
Now let $(y * x)^* \in f^{-1}(B), y \in f^{-1}(B)$, so $f((y * x)^*) \in B$, $f(y) \in B$.
But $f((y * x)^*) = (f(y * x))^* = (f(y) * f(x))^*$ (by Lemma 2.14 (1)).

Therefore $(f(y) * f(x))^* \in B$, $f(y) \in B$, then $f(x) \in B$, since B is a crazy filter, thus $x \in f^{-1}(B)$, that means $f^{-1}(B)$ is a crazy filter in X.

3. The Main Results

In this section, we provide a definitions of complete BCK-filter and complete crazy filter. And, we study its relationship with BCK-filter in BCK-algebra.

Definition 3.1 : A subset F of a bounded BCK-algebra X is said to be complete BCK-filter (c-BCK-filter), if,

(1) $e \in F$

(2) $(x^* * y^*)^* \in F,\ \forall\ y \in F$ implies $x \in F$.

Example 3.2 : Let $X = \{0, 1, 2, 3\}$ and a binary operation $*$ is defined by

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra (see [3]) and $F = \{1, 3\}$ is c-BCK-filter.

Example 3.3 : Let $X = \{0, 1, 2\}$ and a binary operation $*$ is defined by

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra and $F = \{0, 2\}$ is not c-BCK-filter since $(1^* * 0^*)^* = 2 \in F$ and $(1^* * 2^*)^* = 0 \in F$ but $1 \notin F$.

Proposition 3.4 : Every BCK-filter in bounded BCK-algebra X is a c-BCK-filter.

Proof : Let F be a BCK-filter and let $(x^* * y^*)^* \in F,\ \forall\ y \in F$.

Since F is BCK-filter, then $x \in F$. Thus F is c-BCK-filter.

Remark 3.5 : The converse of Proposition 3.4 needs not be true in general as in the Example 3.2 $F = \{1, 3\}$ is c-BCK-filter but it's not BCK-filter, since $(2^* * 1^*)^* = 3 \in F$ but $2 \notin F$.

Corollary 3.6 : In general $\{e\}$, X are trivial c-BCK-filter.

Proposition 3.7 : If X is involutory BCK-algebra then every subset of X contained e is c-BCK-filter.

Proof : Clear by $(x^* * e^*)^* = x$ for all $x \in X$.

Example 3.8 : Let $X = \{0, 1, 2, 3, 4\}$ and a binary operation $*$ is defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	1	1	0	0
4	4	3	2	1	0

It is clear that $(X, *0)$ is an involutory bounded BCK-algebra with unit 4 (see [4]) and $\{4\}, \{0, 4\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \cdots$ etc are c-BCK-filter.

Corollary 3.9 : If X is bounded commutative BCK-algebra then every subset of X contained e is c-BCK-filter.

Proposition 3.10 : If X is bounded BCK-algebra, $|x| > 2$ and $x^* = e$ for all $x \in X$ such that $x \neq e$ then

(1) Every a proper subset F of X contain 0 is not c-BCK-filter

(2) Every a proper subset F of X not contain 0 and contain e is c-BCK-filter.

Proof :

(1) Since $\exists\ x \in X$ such that $(x^* * y^*)^* \in F$, $\forall\ y \in F$ but $x \notin F$.

(2) Since $\forall\ \ x \in X$ such that $(x^* * e^*)^* = 0 \notin F$.

Remark 3.11 : Note that the intersection and union of two c-BCK-filter are not necessary to be c-BCK-filter as shown in the following example.

Example 3.12 : Let $X = \{0, 1, 2, 3, 4\}$ and a binary operation $*$ is defined

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Then $(X, *, 0)$ is bounded BCK-algebra with unit 4 (see [4]). Now let $F_1 = \{0, 1, 4\}$ and $F_2 = \{0, 3, 4\}$, we can show easily that F_1 and F_2 are c-BCK-filter in X. But $F_1 \cap F_2 = \{0, 4\}$ cannot be c-BCK-filter, since $(2^* * 4^*) = 0 \in F_1 \cap F_2$ and $(2^* * 0^*)^* = 4 \in F_1 \cap F_2$, but $2 \notin F_1 \cap F_2$.

So $F_1 \cup F_2 = \{0, 1, 3, 4\}$ cannot be c-BCK-filter, since $(2^* * 4^*)^* = 0 \in F_1 \cup F_2$, $(2^* * 1)^* = 3 \in F_1 \cup F_2$, $((2^* * 3^*)^* = 1 \in F_1 \cup F_2$ and $(2^* * 0)^* = 4 \in F_1 \cup F_2$, but $2 \notin F_1 \cup F_2$.

Proposition 3.13 : Let f be isomorphism from a bounded BCK-algebra X into a bounded BCK-algebra Y, the image of c-BCK-filter is c-BCK-filter.

Proof : Let f be an isomorphism function from bounded BCK-algebra X into bounded BCK-algebra Y and let F be a c-BCK-filter in X.

Then $e_x \in F$ so $(e_x) = e_y \in f(F)$, (by Lemma 2.13).

Now let $(x^* * y^*)^* \in F$, $\forall\ y \in f(F)$, then $f^{-1}((x^* * y^*)^*) \in F, \forall\ f^{-1}(y) \in F$, (since f is onto).

But $f^{-1}((x^* * y^*)^*) = ((f^{-1}(x))^* * (f^{-1}(y))^*)^*$, by (Lemma 2.14(2)). Therefore $((f^{-1}(x))^* * (f^{-1}(y))^*)^* \in F, \forall\ f^{-1}(y) \in F$.

Since F is c-BCK-filter in X, then $f^{-1}(x) \in F$. Thus $x \in f(F)$. That means $f(F)$ is c-BCK-filter in Y.

Proposition 3.14 : Let f be epimorphism from a bounded BCK-algebra X into a bounded BCK-algebra Y, the inverse image of c-BCK-filter is c-BCK-filter.

Proof : Let f be epimorphism from a bounded BCK-algebra X into a bounded BCK-algebra Y and let D be a c-BCK-filter in Y, so $e_y \in D$.

Then $f(e_x) = e_y \in D$ so $e_x \in f^{-1}(D)$, (by Lemma 2.13).

Now let $(x^* * y^*)^* \in f^{-1}(D), \forall\ y \in f^{-1}(D)$, so $f((x^* * y^*)^*) \in D$, $\forall\ f(y) \in D$, since f is onto). But $f((x^* * y^*)^*) = ((f(x))^* * (f(y))^*)^* \in D, \forall\ f(y) \in D$. Then $f(x) \in D$ since D is a c-BCK-filter, therefore $x \in f^{-1}(D)$. Then $f^{-1}(D)$ is c-BCK-filter.

Proposition 3.15 : Assume that X is involutory. Then a nonempty subset F of X is a c-BCK-filter of X if and only if satisfies

(1) $e \in F$

(2) $(y * x)^* \in F, \forall\ y \in F$ implies $x \in F$.

Proof : Let F is c-BCK-filter, then $e \in F$ and let $(y * x)^* \in F, \forall\ y \in F$. Then $(x^* * y^*)^* \in F$, $\forall\ y \in F$ (since X is involutory). Thus $x \in F$ (since F is c-BCK-filter).

Conversely, let $(x^* * y^*)^* \in F, \forall\ y \in F$ since X is involutory. Then $(y * x)^* \in F, \forall\ y \in F$ (since X is involutory). Thus $y \in F$, then F is c-BCK-filter.

Corollary 3.16 : Assume that X is commutative. Then a nonempty subset F of X is a c-BCK-filter of X if and only if satisfies

(1) $e \in F$

(2) $(y * x)^* \in F, \forall\ y \in F$ implies $x \in F$.

Proposition 3.17 : Let X be a bounded BCK-algebra and F be c-BCK-filter of X. If $x \leq y\ \forall\ x \in F$ then $y \in F$.

Proof : Let $x \leq y, \forall\ x \in F$ then $y^* \leq x^*$, since $(y^* * x^*)^* = (0)^* = e \in F$ and F is c-BCK-filter then $y \in F$.

Definition 3.18 : A subset F of a BCK-algebra X is said to be complete crazy-filter (c-Crazy-filter), if,

(1) $e \in F$

(2) $(y * x)^* \in F, \forall\ y \in F$ implies $x \in F$.

Example 3.19 : Let $X = \{0, 1, 2, 3, 4\}$ and a binary operation $*$ is defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra with unit 4 and $F = \{1, 4\}$ is c-crazy-filter.

Proposition 3.20 : Every crazy filter in bounded BCK-algebra X is c-crazy-filter.

Proof : Let F be a crazy filter and let $(y * x)^* \in F, \forall\ y \in F$, since F is crazy filter, then $x \in F$, thus F is c-crazy-filter.

Remark 3.21 : The converse of Proposition 3.20 needs not be true in general as in the example (3.19) $F = \{1, 4\}$ is c-crazy-filter but it's not crazy-filter, since $(1 * 3)^* = 4 \in F$ but $3 \notin F$.

Definition 3.22 : Let F and D be two subset of bounded BCK-algebra X such that $F \subseteq D$, then F is said to be closed with respect to D, if $(x^* * y^*)^* \in D, \forall\ y \in F$, then $(x^* * y^*)^* \in F, x \in X$.

Example 3.23 : Let $X = \{0, 1, 2, 3\}$ and a binary operation $*$ is defined by

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	1	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra with unit 3

(1) Let $F = \{1, 3\}$ and $D = \{0, 1, 3\}$, then F is closed with respect to D.

(2) Let $F = \{0, 3\}$ and $D = \{0, 1, 3\}$, then F is not closed with respect to D, since $(1^* * 3^*)^* = 1 \in D$ and $(1^* * 0^*)^* = 3 \in D$, but $(1^* * 3^*) = 1 \notin F$.

Proposition 3.24 : Let F and D be two subset of involutory bounded BCK-algebra X such that $F \subseteq D$, then F is said to be closed with respect to D, if and only if $(y * x)^* \in D, \forall\ y \in F$, then $(y * x)^* \in F, x \in X$.

Proof : Clear.

Proposition 3.25 : The union of family of closed with respect to D is closed with respect to D.

Proof : Let $\{F_i : i \in \Delta\}$ be a family of closed with respect to D and let $(x^* * y^*)^* \in D, \forall\ y \in \bigcup_{i \in \Delta} F_i$, since $\forall\ i \in \Delta$, F_i is closed with respect to D then $\exists\ j \in \Delta$ such that $(x^* * y^*)^* \in D, \forall\ y \in F_j$, then $(x^* * y^*)^* \in F_j$. Thus $(x^* * y^*)^* \in \bigcup_{i \in \Delta} F_i$, then $\bigcup_{i \in \Delta} F_i$ is closed with respect to D.

Remark 3.26 : Note that the intersection of two closed with respect to D is not closed with respect to D. As it is shown in the following example.

Example 3.27 : Let $X = \{0, 1, 2, 3, 4\}$ and a binary operation $*$ is defined

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	1	1	0	0
2	2	2	0	0	0	0
3	3	3	2	0	2	0
4	4	2	1	1	0	0
5	5	3	4	1	2	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra with unit 5 (see[4]).
Let $D = \{0, 1, 2, 3, 5\}$ and $F_1 = \{0, 1, 3, 5\}, F_2 = \{0, 2, 3, 5\}$ are closed with respect to D, but $F_1 \cap F_2 = \{0, 3, 5\}$ is not closed with respect to D, since $(0 * 0)^* = 5 \in D, (3 * 0)^* = 1 \in D$ and $(5 * 0)^* = 0 \in D$, but $(3 * 0)^* = 1 \notin F_1 \cap F_2$.

Remark 3.28 : If $F \subseteq D$ and F is c-BCK-filter, then D is not in general c-BCK-filter, as in the following example.

Example 3.29 : Let $X = \{0, 1, 2\}$ and a binary operation $*$ is defined by

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

It is clear that $(X, *, 0)$ is a bounded BCK-algebra with unit $2F = \{2\}$ and $D = \{0, 2\}$ F is c-BCK-filter, but D is not c-BCK-filter.

Proposition 3.30 : Let F be c-BCK-filter of bounded BCK-algebra X and $\phi \neq F \subseteq D$. If F is closed with respect to D, then D is c-BCK-filter.

Proof : Let $(x^* * y^*)^* \in D, \forall\ y \in D$. Since $F \subseteq D$, then $(x^* * y^*)^* \in D,\ \forall\ y \in F$. Since F is closed with respect to D, then $(x^* * y^*)^* \in F, \forall\ y \in F$, since F is c-BCK-filter, thus $x \in F$, consequently $x \in D$, then D is c-BCK-filter.

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