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ON CERTAIN SUBCLASSES OF N-FOLD KOEBA TYPE SYMMETRIC FUNCTION

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Abstract

In this present paper, we defined n-fold Koebe type functions

$$F_b(z) = \frac{z}{(1-z^{n+1})^b}, b \geq 0,\ n \in \mathcal{N}$$

in the unit disk. Some basic results of univalence, starlikeness and convexity of the function $F_b(z)$ are obtained in addition with the results of Miller, Mocanu and Reade' and Mocanu[2]. We have also investigated radius of starlikeness and convexity, also other related properties.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions

$$F_b(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the unit disc $\mathcal{U} = \{z : |z| < 1\}$. For some α, we denote the following subclass of $\mathcal{A}$:

$$\mathcal{S} = \{f(z) \in \mathcal{A} : f(z) \text{ is univalent in } \mathcal{U}\} \tag{1.2}$$

$$\mathcal{S}^*(\alpha) = \{f(z) \in \mathcal{A} : f(z) \text{ is univalent and starlike of order } \alpha \text{ in } \mathcal{U}\} \tag{1.3}$$

$$\mathcal{K}(\alpha) = \{f(z) \in \mathcal{A} : f(z) \text{ is univalent and convex of order } \alpha \text{ in } \mathcal{U}\} \tag{1.4}$$

We set $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{K} = \mathcal{K}(0)$. We defined the following new subclasses of $\mathcal{A}$. For some α and β.

$$\mathcal{M}(\alpha,\beta) = \{f(z) \in \mathcal{A} : \mathcal{R}e\left\{(1-\alpha)z\frac{f'(z)}{f(z)} + \alpha\left(1 + z\frac{f''(z)}{f'(z)}\right)\right\} > \beta \text{ in } \mathcal{U}\} \tag{1.5}$$

where $-\infty < \alpha < \infty$ and $\beta < 1$.

$$\mathcal{N}(\alpha,\beta) = \{f(z) \in \mathcal{A} : \mathcal{R}e\left\{(1-\alpha)z\frac{f'(z)}{f(z)} + \alpha\left(1 + z\frac{f''(z)}{f'(z)}\right)\right\} < \beta \text{ in } \mathcal{U}\} \tag{1.6}$$

where $-\infty < \alpha < \infty$ and $\beta > 1$.

Note : These new classes are not necessarily subclasses of $\mathcal{S}$, we put $\mathcal{M}(\alpha) = \mathcal{M}(\alpha, 0)$. The class $\mathcal{M}(\alpha)$ was first introduced by Mocanu[2] who call it the class of α-convex (or α-starlike) functions and later Mocanu, Miller and Reade' studied it and shows that $\mathcal{M}(\alpha)$ is a subclass of $\mathcal{S}^*$ for any real number α and also $\mathcal{M}(\alpha)$ is a subclass of $\mathcal{K}$ for $\alpha \geq 1$. We note that $\mathcal{M}(0) = \mathcal{S}^*$ and $\mathcal{M}(1) = \mathcal{K}$. Mocanu first introduced $\mathcal{M}(\alpha)$ with $\frac{f(z).f'(z)}{z} \neq 0$.

We will investigate some properties of the following Koebe type function

$$F_b(z) = \frac{z}{(1 - z^{n+1})^b}, \quad (b \geq 0, \quad n \in \mathcal{N}). \tag{1.7}$$

1.1 Preliminaries

We will describe some elementary equalities and inequalities without proof which are used in latter discussion. For $n \in \mathcal{N}$, it holds that

$$\mathcal{R}e\left\{\frac{1}{1 - z^n}\right\} = -\frac{1}{2} \quad (|z| = 1, \quad z^n \neq 0) \tag{1.8}$$

$$\mathcal{R}e\left\{\frac{z^n}{1-z^n}\right\} = -\frac{1}{2} \quad (|z|=1, \quad z^n \neq 1) \tag{1.9}$$

$$-\frac{1}{2} < \left\{-\frac{r^n}{1+r^n}\right\} \leq \mathcal{R}e\left\{\frac{z^n}{1-z^n}\right\} \leq \left\{\frac{r^n}{1-r^n}\right\} < \infty \tag{1.10}$$

$$-\infty < \left\{-\frac{r^n}{1-r^n}\right\} \leq \mathcal{R}e\left\{\frac{z^n}{1-z^n}\right\} \leq \left\{\frac{r^n}{1+r^n}\right\} < \frac{1}{2} \tag{1.11}$$

where (1.2) and (1.3) are equivalent relation, so are (1.10) and (1.11) for $|z| \leq r < 1$. Similarly we have

$$-\frac{k}{1-k} < -\frac{kr^n}{1-kr^n} \leq \mathcal{R}e\left\{\frac{kz^n}{1+kz^n}\right\} \leq \frac{kr^n}{1+kr^n} < \frac{k}{1+k}, \text{ when } 0 < k < 1, \tag{1.12}$$

and

$$\frac{k}{1+k} < \frac{kr^n}{1+kr^n} \leq \mathcal{R}e\left\{\frac{kz^n}{1+kz^n}\right\} \leq -\frac{kr^n}{1-kr^n} < -\frac{k}{1-k}, \text{ when } -1 < k < 0 \tag{1.13}$$

where (1.12) and (1.13) are equivalent relation for $|z| \leq r < 1$. Further there holds

$$\frac{1}{1+|k|} < \frac{1}{1+|k|r^n} \leq \mathcal{R}e\left\{\frac{1}{1+kz^n}\right\} \leq \frac{1}{1-|k|r^n} < \infty, \tag{1.14}$$

for $|k| < 1$ and $|z| \leq r < 1$.

2. Univalency, Starlikeness and Convexity

Theorem 2.1 : The function defined by (1.7) is univalent if and only if

$$0 \leq (n+1)b \leq 2 \tag{2.1}$$

Furthermore the condition (2.1) is necessary and sufficient for $F_b(z)$ defined by (1.7) to be a starlike function.

Proof : Since

$$F_b(z) = \frac{z}{(1-z^{n+1})^b}, \quad b \ \geq \ 0. \tag{2.2}$$

Therefore

$$F_b'(z) = \frac{1+[(n+1)b-1]z^{n+1}}{(1-z^{n+1})^{b+1}}$$

$$\Rightarrow |(n-1)b-1| \leq 1 \Rightarrow 0 \leq (n+1)b \leq 2.$$

Conversely, Suppose

$$0 \leq (n+1)b \leq 2.$$

We have

$$\mathcal{R}e\left\{\frac{zF_b'(z)}{F_b(z)}\right\} = \mathcal{R}e\left\{1+(n+1)b\frac{z^{n+1}}{1-z^{n+1}}\right\}$$
$$> 1-(n+1)\frac{b}{2} \in \mathcal{U} \quad \{\text{using} \quad (1.10)\} \tag{2.3}$$

Since $0 \leq (n+1)b \leq 2 \Rightarrow (n+1)\frac{b}{2} \leq 1$. Therefore

$$\mathcal{R}e\left\{\frac{zF_b'(z)}{F_b(z)}\right\} > 0 \tag{2.4}$$

Thus $F_b(z) \in \mathcal{S}^*$ and $F_b(z) \in \mathcal{S}$. Hence

$$F_b(z) \in \mathcal{S} \Leftrightarrow \ 0 \leq (n+1)b \leq 2 \Leftrightarrow \quad \text{for} \quad F_b(z) \in \mathcal{S}^*$$

□

Theorem 2.2 : The function defined by (1.7) is convex if

$$b = 0 \text{ or } b = 1 \ \text{ and } \ n \ = \ 0. \tag{2.5}$$

Proof :

Case-i : If $b = 0$, $F_b(z) = z$ Therefore $F_b(z)$ is univalent in $\mathcal{U}$. Now

$$F_b'(z) = 1, \ F_b''(z) = 0$$

so,

$$\mathcal{R}e\left\{1+z\frac{f_b''(z)}{f_b'(z)}\right\} = \mathcal{R}e\{1+0\} = 1 > 0. \tag{2.6}$$

Therefore $F_b(z) \in \mathcal{K}(\alpha)$.

Case-ii : $b = 1$ and $n = 0$,

$$F_b(z) = \frac{z}{1-z}, \quad F_b'(z) \ = \ \frac{1}{(1-z)^2}, \quad F_b''(z) = \frac{2}{(1-z)^3}$$

$$\mathcal{R}e\left\{1+z\frac{f_b''(z)}{f_b'(z)}\right\} = \mathcal{R}e\left\{1+z.\frac{2z}{(1-z)}\right\} > \mathcal{R}e\left\{1-\frac{2}{(2)}\right\} = 0 \tag{2.7}$$

using (1.10).

So, $F_b(z) \in \mathcal{K}(\alpha)$. We must show that $F_b(z) \notin \mathcal{K}(\alpha)$ in any other cases. We have,

$$\left\{1+z\frac{f_b''(z)}{f_b'(z)}\right\} = 1-b(n+1)+\frac{(n+1)(b+1)}{1-z^{n+1}}-\frac{(n+1)}{1+[b(n+1)-1]z^{n+1}}$$
$$= g(\xi) \tag{2.8}$$

so,

$$g(\xi) = 1 - b(n+1) + \frac{(n+1)(b+1)}{1-\xi} - \frac{(n+1)}{1+[b(n+1)-1]\xi} \tag{2.9}$$

where $\xi = z^{n+1}$ (say).

$$g(\xi) = \left\{1 + z\frac{f_b''(z)}{f_b'(z)}\right\}, \quad \xi = z^{n+1}, \quad \text{Then} \quad z \in \mathcal{U} \Leftrightarrow \xi \in \mathcal{U}. \tag{2.10}$$

The following cases will arise.

Case i : When $b(n+1) < 2$, then $F_b(z) \notin \mathcal{S}$. So $F_b(z) \notin \mathcal{K}$.

Case ii : When $b(n+1) < 2$, then (2.9) becomes

$$g(\xi) = -1 + \frac{n+3}{1-\xi} - \frac{n+1}{1+\xi} \tag{2.11}$$

If we put $1 + \xi = \rho$ i.e. $\xi = -1 + \rho$, $\rho > 0$, a small number,
it follows that $\xi \in \mathcal{U}$ and $\mathcal{R}e\{g(\xi)\} \to -\infty$ as $\rho = 0$.
Hence there exist $\xi \in \mathcal{U}$ such that $\mathcal{R}e\{g(\xi)\} < 0$. So, $F_b(z) \notin \mathcal{K}$

Case iii : When $1 < b(n+1) < 2$, we have $0 < k < 1$ (*since* $k = b(n+1) - 1$) then from (2.9)

$$g(\xi) = -k + \frac{k+n+2}{1-\xi} - \frac{n+1}{1+k\xi} \tag{2.12}$$

Then

$$g(-1) = -k + \frac{k+n+2}{1+1} - \frac{n+1}{1-k} < 0. \tag{2.13}$$

On the other hand, since $\mathcal{R}e\{g(\xi)\}$ is continuous at $\xi = -1$, there exist $\xi_o \in \mathcal{U}$ sufficiently close to $\xi = -1$ such that $\mathcal{R}e\{g(\xi)\} < 0$. So $F_b(z) \notin \mathcal{K}$.

Case iv : When $b(n+1) = 1$, then from (2.9)

$$g(\xi) = \frac{n+2}{1-\xi} - \frac{n+1}{1} = \frac{1+(n+1)\xi}{1-\xi} \tag{2.14}$$

$$g(-1) = -\frac{n}{2} < 0, \quad \text{for} \quad n \in \mathcal{N} \tag{2.15}$$

So $F_b(z) \notin \mathcal{K}$.

Then, $F_b(z)$ in every case apart from (2.5). This completes the proof. □

3. Radii of Univalence, Starlikeness and Convexity

Theorem 3.1 : The function $F_b(z)$ defined by (1.7) with $(n+1)b - 1 > 1$ is univalent and starlike for

$$|z| < \left\{\frac{1}{(n+1)b-1}\right\}^{\frac{1}{n}} \tag{3.1}$$

This bound is the best possible for univalence and starlikeness.

Proof :

$$\mathcal{R}e\left\{z\frac{f_b'(z)}{f_b(z)}\right\} = \mathcal{R}e\left\{1+(n+1)b\left(\frac{z^{n+1}}{1-z^{n+1}}\right)\right\}$$

$$\geq \left\{1+(n+1)b\left(-\frac{r^{n+1}}{1+r^{n+1}}\right)\right\}, \text{ for } |z|\leq r<1 \tag{3.2}$$

$$= \frac{1-[(n+1)b-1]r^{n+1}}{1+r^{n+1}}, \text{ for } |z|\leq r<1$$

So, $F_b(z)$ is univalent and starlike in the disk

$$|z| < \left\{\frac{1}{(n+1)b-1}\right\}^{\frac{1}{n+1}} \tag{3.3}$$

Because of $F_b'(z_0)=0$, $z_0^{n+1} = -\frac{1}{(n+1)b-1}$. This is the best possible bound. □

Theorem 3.2 : The function $F_b(z)$ defined by (1.7) with $(n+1)b-1>0$ is univalent and convex for

$$|z| < \sqrt[n+1]{t_0} \tag{3.4}$$

where t_0 is the smallest positive roots of the equation

$$h_1(t) = k^2t^2-(3k+nk+n+1)t+1=0, \ k=(n+1)b-1. \tag{3.5}$$

The radius of disk is given by (3.4) is best possible.

In particular $t_0 = \frac{1}{n+1}$ when $(n+1)b=1$ and $t_0=(n+2)-\sqrt{(n+1)(n+3)}$ where $(n+1)b=2$.

Proof : We use

$$g(\xi) = -k+\frac{k+n+2}{1-\xi}-\frac{n+1}{1+k\xi}, \quad k=(n+1)b-1, \quad k\geq 0 \tag{3.6}$$

Then

$$\begin{aligned}\mathcal{R}e\{g(\xi)\} &= -k+\frac{k+n+2}{1+t}-\frac{n+1}{1-kt}<0, \quad \text{(using 1.14)} \\ &= \frac{k^2t^2-(3k+nk+n+1)t+1}{(1-kt)(1+t)}.\end{aligned} \tag{3.7}$$

For $|\xi|\leq t<1$ when $0\leq k\leq 1$ and for $|\xi|\leq t<\frac{1}{k}$ when $k>1$. Therefore if t_0 is the smallest positive root of (3.5) then $\mathcal{R}e\{g(\xi)\}>0$ for $|\xi|<t_0$. We can verify that $t_0\leq 1$ when $0\leq k<1$ and $t_0\leq\frac{1}{k}$ when $k>1$. In fact

Case i : When $0\leq k<1$, $h_1(0)=1>0$ and

$$h_1(1)=k^2-(3k+nk+n)=k(k-3)-n(k+1)\leq 0,$$

Therefore

$$0 < t_0 \leq 1$$

.

Case ii : When $k > 1$, $h(0) = 1 > 0$. and $h_1(\frac{1}{k}) = -(n+1) - \frac{(n+1)}{k} < 0$. Therefore

$$0 < t_0 \leq \frac{1}{k}$$

Thus $F_b(z)$ is convex in $|z| < (n+1)\sqrt{t_0}$. Therefore this bound is the best possible. □

Theorem 3.3 : The function $F_b(z)$ defined by (1.7) with $-1 < \{(n+1)b - 1\} < 0$ is convex for

$$|z| < \sqrt[n+1]{t_1} \tag{3.8}$$

where t_1 is the smallest positive root of the equation.

$$h_2(t) = k^2t^2 + (3+n)kt + (n+1)t + 1 = 0, \; k = (n+1)b - 1. \tag{3.9}$$

The radius of disk is given by (3.8) is not the best possible.

Proof : $-1 < (n+1)b - 1 < 0$ i.e. $-1 < k < 0$

$$\mathcal{R}e\{g(\xi)\} = k + \frac{-k+n+2}{1+t} - \frac{n+1}{1+kt} = \frac{k^2t^2 + (3+n)kt - (n+1)t + 1}{(1+kt)(1+t)} > 0. \tag{3.10}$$

For $|\xi| \leq t_1$.

$$h_2(t) = k^2t^2 + (3+n)kt - (n+1)t + 1$$

Then

$$h_2(0) = 1 \;\text{ and }\; h_2(1) = k^2 + (3+n)k - (n+1) + 1 < 0$$

Therefore

$$0 < t_1 \leq 1$$

Thus $F_b(z)$ is convex in $|z| < (n+1)\sqrt{t_1}$. Therefore this bound is the best possible. □

4. Relation Between $f_b(z)$ and the Class $\mathcal{M}(\alpha,\beta)$ and $\mathcal{N}(\alpha,\beta)$

Theorem 4.1 : Corresponding to the function $F_b(z)$ defined by (1.7), let

$$\beta_1 = \frac{[2-(n+1)b](b-\alpha)}{2b} \qquad \beta_2 = \frac{[2-(n+1)b]^2-(n+1)^2\alpha b}{2[2-(n+1)b]} \tag{4.1}$$

Case I: If $\alpha \geq 0$ then

$$i.F_b(z) \in \mathcal{M}(\alpha,\beta_1) \text{ for } 0 < (n+1)b \leq 1, \tag{4.2}$$

$$ii.F_b(z) \in \mathcal{M}(\alpha,\beta_2) \text{ for } 1 \leq (n+1)b < 2,$$

Case II: If $\alpha \leq 0$ and $b+\alpha > 1$ then

$$i.F_b(z) \in \mathcal{M}(\alpha,\beta_2) \text{ for } 0 < (n+1)b \leq 1, \tag{4.3}$$

$$ii.F_b(z) \in \mathcal{M}(\alpha,\beta_1) \text{ for } 1 \leq (n+1)b < 2,$$

Case III: If $\alpha \leq 0$ and $b+\alpha < 1$ then

$$i.F_b(z) \in \mathcal{N}(\alpha,\beta_2) \text{ for } 0 < (n+1)b \leq 1, \tag{4.4}$$

$$ii.F_b(z) \in \mathcal{M}(\alpha,\beta_2) \text{ for } 1 \leq (n+1)b < 2,$$

Remark 4.1 : The parameter is the best value for any case.

Remark 4.2 : $\beta_1 = \beta_2 \Leftrightarrow \alpha[(n+1)b-1] = 0$.

Proof : We put

$$\begin{aligned} w(z) &= \mathcal{R}e\left\{(1-\alpha)z\frac{f_b'(z)}{f_b(z)} + \alpha\left(1+z\frac{f_b''(z)}{f_b'(z)}\right)\right\} \\ &= 1+(n+1)(b+\alpha)\left(\frac{z^{n+1}}{1-z^{n+1}}\right) + (n+1)\alpha\left(\frac{[(n+1)b-1]z^{n+1}}{1+[(n+1)b-1]z^{n+1}}\right) \end{aligned} \tag{4.5}$$

Case I, i : If $\alpha \geq 0,\ 0 < [(n+1)b] \leq 1$ i. e. $-1 < [(n+1)b-1] \leq 0$

$$\begin{aligned} w(z) &> 1-(n+1)\frac{(b+\alpha)}{2} + (n+1)\alpha\left(\frac{[(n+1)b-1]}{1+[(n+1)b-1]}\right), \quad z \in \mathcal{U} \\ &= 1-(n+1)\frac{b}{2} + (n+1)\frac{\alpha}{2} - \frac{\alpha}{b} \\ &= \beta_1 \end{aligned} \tag{4.6}$$

We must have $\beta_1 < 1$ because of $w(0) = 1$.

So $f_b(z) \in \mathcal{M}(\alpha,\beta_1)$ for $0 < (n+1)b \leq 1$.

ii-If $\alpha \geq 0, \quad 1 \leq [(n+1)b] < 2$ i. e. $0 \leq [(n+1)b-1] < 1$,

$$w(z) > 1-(n+1)\frac{(b+\alpha)}{2}-(n+1)\alpha\left(\frac{[(n+1)b-1]}{1+[(n+1)b-1]}\right) \quad z \in \mathcal{U}, \tag{4.7}$$

using (1.10) and (1.12)

$$w(z) = \frac{[2-(n+1)b]^2-(n+1)^2\alpha.b}{2[2-(n+1)b]} = \beta_2 \tag{4.8}$$

We must have $\beta_2 < 1$ because of $w(0) = 1$.

So $f_b(z) \in \mathcal{M}(\alpha, \beta_2)$ for $0 \leq [(n+1)b-1] < 1$ i.e. $1 \leq [(n+1)b] < 2$.

Case II, i : If $\alpha \leq 0, \ b+\alpha > 1$ for $0 < [(n+1)b] \leq 1$ i.e. $-1 < [(n+1)b-1] \leq 0$

$$\begin{aligned} w(z) &> 1-(n+1)\tfrac{(b+\alpha)}{2}-(n+1)\alpha\left(\tfrac{[(n+1)b-1]}{1+[(n+1)b-1]}\right), \quad z \in \mathcal{U} \\ &= \beta_2 \end{aligned} \tag{4.9}$$

Thus $\beta_2 < 1$ because of $w(0) = 1$ So $f_b(z) \in \mathcal{M}(\alpha, \beta_2)$ for $0 < (n+1)b \leq 1$.

ii : If $\alpha \leq 0, \ b+\alpha > 1 \ \ 1 \leq [(n+1)b] < 2$ i. e. $0 \leq [(n+1)b-1] < 1$

$$\begin{aligned} w(z) &> 1-(n+1)\tfrac{(b+\alpha)}{2}+(n+1)\alpha\left(\tfrac{[(n+1)b-1]}{1+[(n+1)b-1]}\right), \\ &= \beta_1 \end{aligned} \tag{4.10}$$

Thus $\beta_1 < 1$ as $w(0) = 1$

So $f_b(z) \in \mathcal{M}(\alpha, \beta_1)$ for $0 \leq [(n+1)b-1] < 1$ i. e. $1 \leq [(n+1)b] < 2$.

Case III, i : If $\alpha \leq 0, \ b+\alpha < 1$ and $0 < [(n+1)b] \leq 1$ i.e. $-1 < [(n+1)b-1] \leq 0$.

$$\begin{aligned} w(z) &< 1-(n+1)\tfrac{(b+\alpha)}{2}-(n+1)\alpha\left(\tfrac{[(n+1)b-1]}{1-[(n+1)b-1]}\right), \quad z \in \mathcal{U} \\ &= \beta_2 \end{aligned} \tag{4.11}$$

Thus $\beta_2 > 1$ because of $w(0) = 1$.

So $f_b(z) \in \mathcal{N}(\alpha, \beta_2)$ for $0 < (n+1)b \leq 1$.

ii:If $\alpha \leq 0, \ b+\alpha < 1$ for $1 \leq [(n+1)b] < 2$ i. e. $0 \leq [(n+1)b-1] < 1$

$$\begin{aligned} w(z) &> 1-(n+1)\tfrac{(b+\alpha)}{2}-(n+1)\alpha\left(\tfrac{[(n+1)b-1]}{1-[(n+1)b-1]}\right), \\ &= \beta_2 \end{aligned} \tag{4.12}$$

Thus $\beta_2 < 1$ because of $w(0) = 1$.

So $f_b(z) \in \mathcal{M}(\alpha, \beta_2)$ for $1 < (n+1)b \le 2$. □

Corollary 4.1 : The function $f_b(z)$ defined by (1.7) is in $\mathcal{M}(\alpha)$ if it satisfies one of the following three conditions

$$0 \le \alpha \le b \le \frac{1}{n+1} \tag{4.13}$$

$$0 \le \alpha \le \frac{[2-(n+1)b]^2}{(n+1)^2 b}, \quad \frac{1}{n+1} \le b < \frac{2}{n+1} \tag{4.14}$$

$$-b < \alpha \le 0 \tag{4.15}$$

Proof : (4.13) follows from case I (i)

(4.14) follows from case I (ii) as Theorem 6.

(4.15) follows from case II (i)and case II (ii) □

Corollary 4.2 : The function $f_b(z)$ in corollary 4.1 is univalent and starlike of order

$$1 - \frac{(n+1)b}{2} \in \mathcal{U} \tag{4.16}$$

Proof : If $\alpha = 0$ in case II, the function $f_b(z)$ defined by (1.7) is univalent and starlike of order $1 - \frac{(n+1)b}{2}$ Since

$$\mathcal{R}e\left(\frac{zf_b'(z)}{f_b(z)}\right) > 1 - \frac{(n+1)b}{2} \tag{4.17}$$

□

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