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PAIRED NEIGHBOURHOOD SET ON INTERVAL GRAPHS

Y. LAKSHMI NAIDU[1] **AND SARANYA C. R.**[2]
[1] Department of Mathematics,
Sri Sathya Sai Institute of Higher Learning,
Anantapur Campus, A.P. 515001, India
[2] Department of Mathematics,
Smt. Eswaramma High School,
Prasanthinilayam, A. P. 515134, India

Abstract

In this paper we study the paired neighbourhood set problem on interval graphs. Given an interval graph G with end points sorted, we give an $0(n+m)$ algorithm to find a paired neighbourhood set, where n and m denote the number of vertices and number of edges respectively.

1. Introduction

Let $G(V, E)$ be a graph. The closed neighbourhood of a vertex $v \in V$ in G is defined as v and the set of vertices that are adjacent to v in G. The closed neighbourhood is denoted by $N[v]$. A set S of vertices in G is called a neighbourhood set of G if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph induced by $N[v]$. A neighbourhood set with minimum

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cardinality is called a minimum neighbourhood set. A subset S of V is called a paired neighbourhood set, if S is a neighbourhood set and the induced subgraph on S has a perfect matching.

A graph $G(V, E)$ is called an intersection graph for a family $\mathcal{F}$ of a non empty set, if there is a one-to-one correspondence between $\mathcal{F}$ and V such that two sets in $\mathcal{F}$ have non empty intersection if and only if their corresponding vertices in V are adjacent. If $\mathcal{F}$ is a family of intervals on a real line, then G is called an interval graph for $\mathcal{F}$. Each interval is represented as a vertex and there is an edge between two vertices if the corresponding intervals intersect each other.

Interval graphs have found applications in a wide range of fields such as genetics, scheduling, computer science etc., they have been studied by many researchers [1, 2, 3]. Neighbourhood and Paired neighbourhood set problems have been researched in [4,5,6,7]. In the following sections we present an algorithm to find a minimum paired neighbourhood set of an interval graph and also give the proof of correctness of the algorithm.

2. Preliminaries

Let $I = \{1, 2, 3, \cdots, n\}$ be a family of intervals on real line. Each interval in I is represented by $[a_i, b_i]$ for $i = 1, 2, \cdots, n$. Here a_i is called the left endpoint and b_i is called the right endpoint of interval i. The intervals are labelled in increasing order of their right endpoints. This ordering is possible and can be obtained in $0(n + m)$ time, where n is the number of vertices and m is the number of edges as in [2].

For each interval i we define $IEB(i)$ as the set of intervals whose right endpoints are less than a_i, that is set of intervals ending before i. $\max a(IEB(i))$ is the interval with largest left endpoint ending before i. $mate(i)$ be the interval that intersects i with minimum left endpoint and $Pred(i)$ is the interval that intersects i with minimum left endpoint and whose right endpoint is less than b_i.

Following the above definitions, we present an algorithm MPN to find the minimum paired neighbourhood of an interval graph $G[I]$.

Let $I = \{1, 2, 3, \cdots, n\}$ be a family of intervals on real line, we introduce two intervals $n + 1$ and $n + 2$ with $a_{n+1} = 2n + 1, a_{n+2} = 2n + 2, b_{n+1} = 2n + 3$ and $b_{n+2} = 2n + 4$. Let I_p be the set of intervals obtained by augmenting I with the two intervals $n+1$ and $n+2$. The following algorithm finds the minimum paired neighbourhood set of interval graph $G[I]$.

3. Algorithm MPN

Input : Family of intervals $I_p = \{1, 2, 3, \cdots, n, n+1, n+2\}$

Output : A minimum paired neighbourhood set MPN of $G[I_p]$.

Step 1 : Start

Step 2 : Find $\max a(IEB(j))$ for all $j \in I_p$.

Step 3 : Find $mate(j)$ for all $j \in I_p$.

Step 4 : Find $Pred(j)$ for all $j \in I_p$

Step 5 : $MPN(0) = 0$

Step 6 : For $j = 1$ to $n + 2$ do

m be the interval with maximum left endpoint in the set

$a(IEB(\min(j, mate(j))))$

$m = \max a(IEB(\min(j, mate(j))))$

Step 7 : Let $k = pred(m)$

Step 8 : $MPN(j) = \{mate(j), j) \cup MPN(k)$.

Step 9 : End.

Output : $MPN(n + 2)$.

4. Proof of Correctness

We prove that the algorithm is correct in the sense that the output $MPN(n + 2)$ is minimum paired neighbourhood of $G[I_p]$.

Lemma 1 : The intervals whose left endpoints are between j and $mate(j)$ intersects j or $mate(j)$.

Proof : Since the intervals are labelled in the increasing order of their right endpoints, the proof of the statement follows immediately.

Lemma 2 : Let j and m be two intervals such that $m = \max a(IEB(\min(j, mate(j))))$ then j and m do not intersect.

Proof : By definition $m = \max a(IEB(\min(j, mate(j))))$, two different cases arise here.

Case (i) : $\min(j, mate(j)) = j$.

In this case, by definition m is an interval which ends before j and hence they do not intersect.

Case (ii) : $\min(j, mate(j)) = mate(j)$.

By definition of $mate(j)$, the left endpoint of $mate(j)$ is the minimum among all intervals which intersect j. In this case too m is an interval which ends before j. Hence they do not intersect.

Theorem 1 : For any interval j, the set $\{j, mate(j), k, mate(k)\}$ where $k = pred(m)$ and $m = \max a(IEB(\min(j, mate(j))))$ forms a paired neighbourhood set for the induced graph of the intervals from k to j.

Proof : Let $H = G[k, \cdots, j]$ and $H' = G[N[j, mate(j), k, mate(k)]]$.

The proof is done if we show that all the vertices and all the edges in H are contained in H'.

By using the definition of $IEB(j)$ and the definition of k, if there is an interval which does not belong to $N[j, mate(j), k, mate(k)]$, then k cannot be the $pred(m)$ which is a contradiction since $k = pred(m)$. Thus $N[j, mate(j), k, mate(k)]$ contains all the intervals from k to j.

Hence, all the vertices of H are in H'.

Now, we must show that all the edges of H are in H'. Suppose there is an edge which belongs to H but is not in H'. Then $(x, y) \notin H'$ implies that both x and y does not belong to $N[j]$ or $N[mate(j)]$ or $N[k]$ or $N[mate(k)]$. Without loss of generality let $x \in N[j, mate(j)]$ and $y \in N[k, mate(k)]$.

Then by Lemma 2, it is clear that $x \notin N[k, mate(k)]$. Then by the definitions of IEB and $pred(j)$, k cannot be $pred(j)$, which is a contradiction.

Hence $x \in N[k, mate(k)]$, therefore $(x, y) \in H'$.

$$\therefore \quad H \subseteq H'.$$

Theorem 2 : The algorithm MPN produces the minimum paired neighbourhood set of $G(I)$.

Proof : From Lemma 2 and Theorem 1, it is clear that MPN produces a paired neighbourhood set of $G[I]$. We have to show that the paired neighbourhood set produced by MPN is the minimum paired neighbourhood set of $G[I_p]$.

Let $m_t = \max a(\min(I_t, mate(I_t))$ and $I_{t-1} = pred(m_t)$.

Suppose $MPN = \{I_1, mate(I_1), \cdots, I_t, mate(I_t)\}$. Therefore $|MPN| = 2t$.

Let S be any minimum paired neighbourhood of $G[I_p]$, then it is enough to show that

$|MPN| \leq |S|$. Consider any two consecutive pairs say $(I_j, mate(I_j))$ and $(I_{j+1}, mate(I_{j+1}))$, then by Theorem 1, $H = G[I_j \cdots I_{j+1}]$is contained in $H' = G[N[I_j, mate(I_j),$ $I_{j+1}, mate(I_{j+1})]]$.
Suppose $(I_m, mate(I_m))$ is a pair of intervals in I such that $G[I_m, mate(I_m)] \not\subseteq H'$. Then since S is minimum neighbourhood of $G[I]$, there is a pair in induced graph of S, such that $G[I_m, mate(I_m)] \cup H'$ is covered by it. Thus for any two consecutive pairs in MPN there is at least one pair in S. As there are t such pairs in MPN, there must be atleast t such pairs in S. Thus $|S| \geq 2t$.

$$\therefore \quad |MPN| \leq |S|.$$

Hence MPN is a minimum paired neighbourhood set.
Time complexity :
Theorem 3 : Given an interval family labelled in increasing order of their right endpoints, the minimum paired neighbourhood $G[I]$ of can be found in $0(n+m)$ time.
Proof : By an algorithm given by Chang [2], to find max $a(IEB(j))$ for all $j \in I_p$ given in step 1, it would take $0(n)$ time. The time taken to perform step 2 and step 3 is at most $0(m)$. Since all the values used in the for loop are already available, to find m given in the loop takes $0(n)$ time.
Thus overall time complexity to find minimum paired neighbourhood is $\mathbf{0(n+m)}$.

5. Conclusion

Given an interval graph with endpoints sorted, we have presented an $0(n+m)$ time algorithm to solve the minimum paired neighbourhood problem on interval graphs. The result can be extended to find minimum paired neighbourhood of circular-arc graphs.

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