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GENERALIZATION OF BANACH, KANNAN AND CHATTERJEE FIXED POINT THEOREM

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Abstract

The Banach contraction principle [2] is the first important result on fixed points for contractive type mappings. In 1968 Kannan [4] and in 1972 Chatterjee [3] gives some interesting results on fixed point. Our aim in this paper to discuss about fixed point theory and also established fixed point theorem in complete metric space, which is a new generalized result in fixed point theory.

1. Introduction

Stefan Banach (1892-1945) was the famous Polish Mathematician who was one of the founder of fixed point theory. In 1922 Banach [2] prove fixed point important result on contractive type mappings. Generalization of Banach fixed point theorem has been a heavily investigated branch of research. So far, according to importance and simplicity, many authors have obtained, interesting extensions and generalization of the Banach contraction principle. In 1968 Kannan [4] and in 1972 Chatterjee [3] studied contractive mappings which gives unique fixed point on complete metric space. Now our aim in this paper to established fixed point theorem which generalized many results of above mathematicians.

2. Definition and Preliminaries

Definition 2.1 (L. Peeler [5]) : Let X be any non-empty set. A metric on X is a mapping $\varrho : X \times X \to R$ which satisfies the following properties, for all $x, y, z \in X$.

(M1) $\varrho(x, x) = 0$

(M2) $\varrho(x, y) = 0 \Rightarrow x = y$ (Identity)

(M3) $\varrho(x, y) = \varrho(y, x)$ (Symmetry)

(M4) $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$ (Triangle inequality)

The set X together with a metric ϱ on it is metric space.

Example 2.1 : Let $C[0, 1]$ denote the collection of all real valued continuous function defined on [0,1]. We defined the norm of $f \in [0, 1]$ by,

$$\|f\| = \sup\{|f(x)|/x \in [0, 1]\}.$$

Let ϱ be defined by $\varrho(x, y) = \|f - g\|$ then ϱ is a metric for $C[0, 1]$.

Definition 2.2 (L. Peeler [5]) : Let X be a non-empty set and f be self-map on X, then a point $x \in X$ is called a fixed point of f if $f(x) = x$.

Example 2.2 : If f is self-map on real numbers by $f(x) = x^2 - 3x + 4$. Then 2 is a fixed point of f because $f(2) = 2$.

Definition 2.3 (Singh [9]) : Let f be self-map on metric space X then f is said to be contraction mapping if,

$$\varrho(f(x), f(y)) \leq \gamma\varrho(x, y), \quad \text{for all} \quad x, y \in X. \tag{2.1}$$

for some real number γ such that $0 < \gamma < 1$.

Remark 2.1 : The mapping f is contraction mapping is the distance between the image of any two points is less than between their pre-images.

Remark 2.2 : The f is contraction on X then it is continuous on X.

Example 2.3 : The function $f : R \to R$ defined by $f(x) = \frac{x}{2} + 1$ is contraction with real number $\gamma = \frac{1}{2} \in (0, 1)$.

Definition 2.4 (L. Peeler [5]) : A metric space X is said to be complete if every Cauchy sequence in X converges.

Example 2.4 : The space of $C[0,1]$ of all continuous real valued function on $[0,1]$ with metric ϱ defined by,

$$\varrho(f,g) = \sup\{|f(x) - g(x)|/x \in [0,1]\}$$

is complete metric space.

Definition 2.5 (Harsain, Parvaneh, Samet and Vetro [10]) : Let (X,d) be metric space self-map f on X is called JS-contraction if three exist $T \in \psi$ and non-negative numbers a,b,c,d with $a+b+c+2d < 1$ such that,

$$T(\varrho(fx,fy)) \leq T[(\varrho(x,y))]^a T[(\varrho(x,fx))]^b [T(\varrho(y,fy))]^c [T(\varrho(x,fy)) + \varrho(T(y,fx))]^d \text{ for all } x,y \in X$$

Where ψ is a set of function $T = [0,\infty] \to [1,\infty]$ satisfying the following condition

1. T is increasing and $T(d) = 1$ iff $d = 0$.
2. For each sequence $\{d_n\} \subset (0,\infty)$

$$\lim_{n\to\infty} T(d_n) = 1 \text{ iff } \lim_{n\to\infty} d_n = 0.$$

3. $\exists b \in (0,1)$ and $l \in (0,\infty)$ such that $\lim_{d\to 0^+} \frac{T(d)-1}{d^b} = l$.
4. $T(a+b) \leq T(a)T(b)$ for all $a,b > 0$.

3. Some Fixed Point Theorems On Complete Metric Space

To the point of view to find out fixed point we study the following theorems.

Theorem 3.1 (Banach [2]) : $f : X \to X$, where (X,ϱ) is complete metric space satisfying (2.1) will have a unique fixed point.

Proof : Let satisfies (2.1) on complete metric space X.

$$\text{Let } \quad x_0 \in X \text{ and } x_n \in f_{x_{n-1}} = f_{x_0}^n \text{ for any } n \in I. \tag{3.1}$$

By (2.1) we get,

$$\varrho(f_x^n, f_y^n) \leq \gamma^n \varrho(x,y), \quad \forall \ x,y \in X \text{ and } \forall \ n \in N. \tag{3.2}$$

If m, n are positive integers such that $m > n$.
Let $m = n + p$ where p is positive integer.
Then by triangle inequality, (3.1) and (3.2) we get,

$$\varrho(x_n, x_m) \leq \gamma^n \varrho(x, y) \frac{1}{1-\gamma} \tag{3.3}$$

$\because \quad \gamma^n \to 0$ as $n \to \infty$.
$\therefore \varrho(x_n, x_m)$ can be made arbitarary small by taking n, sufficiently large.
$\therefore \quad \langle x_n \rangle$ is Cauchy Sequence.
$\therefore \quad X$ is complete.
$\therefore \quad f$ is continuous.

$$\therefore \quad f_x = \lim_{n\to\infty} f_{x_n} = \lim_{n\to\infty} (x_n + 1) = x \tag{3.4}$$

$\therefore \quad f$ has a fixed point.
Let $y \in X$ such that $y \neq x$ and $T_y = y$. Then,

$$\varrho(x, y) = \varrho(f_x, f_y) \leq \gamma \varrho(x, y) \Rightarrow 1 \leq \gamma$$

which is a contradiction to $0 < \gamma < 1$.
$\therefore \quad x$ is the only point in X such that, $f(x) = x$.
The slight generalization Banach theorem is given by Kannan, Chatterjee and Rhoades as follows.

Theorem 3.2 (Kannan [4]) : Let f be a self-map of a complete metric space X such that,

$$\varrho(f_x, f_y) \leq \beta[\varrho(f_x, x) + \varrho(f_y, y)] \tag{3.5}$$

for a number $\beta, 0 < \beta < \frac{1}{2}$ for each $x, y \in X$. Then f has a unique fixed point.

Theorem 3.3 (Chatterjee [3]) : Let f be a self-map of a complete metric space X such that,

$$\varrho(f_x, f_y) \leq \alpha[\varrho(f_x, x) + \varrho(f_y, y)] \tag{3.6}$$

for a number of $\alpha, 0 < \alpha < \frac{1}{2}$ for each $x, y \in X$. Then f has a unique fixed point.

Theorem 3.4 (Rhoades [6]) : Let f be self-map on metric space X and let exists a number $0 \leq h < 1$, for all $x, y \in X$ such that,

$$\varrho(f(x), f(y)) \leq h \ \max(\varrho(x, f(y)) + \varrho(y, f(x))).$$

Then f has unique fixed point.

Theorem 3.5 (Shujun Jiang, Zhilong Li and Bosko Dmjanovic [8]) : Let (X, ϱ) be a complete metric space and $f : X \to X$ be a JS-contraction with $T \in \psi_2$. Then f has a unique fixed point X.

By using definition (2.1) and from theorem 3.1 , 3.2 and 3.3 we generalized our main result.

4. Main Result

Theorem 4.1 : Let f be a self-mapping of a complete metric space X. For real numbers $0 \leq \alpha < \beta < \gamma < 1$ with $\alpha \leq \frac{2}{5}$ and $\beta \leq \frac{3}{5}$. Also $x \neq y \in X$ at least one of the following conditions are satisfied.

(a) $\varrho(f(x), f(y)) \leq \gamma\varrho(x, y)$

(b) $\varrho(f(x), f(y)) \leq \beta(\varrho(x, f(x)) + \varrho(y, f(y)))$.

(c) $\varrho(f(x), f(y)) \leq \alpha(\varrho(x, , f(y)) + \varrho(y, f(x)))$.

Then f has a unique fixed point.

Proof : Consider real numbers,

$$0 \leq \alpha < \beta < \gamma < 1 \;\text{ with }\; a \leq \frac{2}{5} \;\text{ and‘ }\; \beta \leq \frac{3}{5}.$$

Choose $x_0 \in X$.

Let $x = f^n(x_0)$ where $n \geq 0$ be fixed integer and $y = f^{n+1}(x_0)$.

Suppose $x \neq y$.

For these two points condition (a) is satisfied then,

$$\begin{aligned} \varrho(f^{n+1}(x_0), f^{n+2}(x_0)) &\leq \gamma\varrho(f^n(x_0), f^{n+1}(x_0)) \\ &\leq 2\gamma\varrho(f^n(x_0), f^{n+1}(x_0)). \end{aligned}$$

If for x, y condition (b) is verified then,

$$\varrho(f^{n+1}(x_0), f^{n+2}(x_0)) \leq \beta(\varrho(f^n(x_0), f^{n+1}(x_0)) + \varrho(f^{n+1}(x_0), f^{n+2}(x_0)))$$

which implies

$$\begin{aligned} \varrho(f^{n+1}(x_0), f^{n+2}(x_0)) &\leq 2\beta(\varrho(f^n(x_0), f^{n+1}(x_0))) \\ &\leq 2\gamma\varrho(f^n(x_0), f^{n+1}(x_0)). \end{aligned}$$

In case condition (1) is satisfied,

$$
\begin{aligned}
\therefore\quad \varrho(f^{n+1}(x_0), f^{n+2}(x_0)) &\leq \alpha(\varrho(f^n(x_0), f^{n+2}(x_0)) + \varrho(f^{n+1}(x_0), f^{n+1}(x_0)))\\
&\leq \alpha\varrho(f^n(x_0), f^{n+2}(x_0))\\
&\leq \alpha(\varrho(f^n(x_0), f^{n+1}(x_0)) + \varrho(f^{n+1}(x_0), f^{n+1}(x_0)))\\
&\leq 2\alpha(f^n(x_0), f^{n+1}(x_0))\\
&\leq 2\gamma(f^n(x_0), f^{n+1}(x_0)).
\end{aligned}
$$

This inequality true for every n.

$\therefore$ $\{f^n(x_0)\}_{n=0}^{\infty}$ is a Cauchy sequence.

$\therefore$ Converges to some point $z \in X$.

Now, prove that, z is a fixed point of f.

Suppose $f(z) \neq z$.

Consider the ball $B = \left\{x \in X/\varrho(x, z) \leq \frac{1}{4}\varrho(z, f(z))\right\}$.

There exists a number N such that, $f^n(x_0) \in B$ for each $n \geq N$.

Let $x = f^N(x_0)$ and $y = z$ we get,

$$\varrho(f^{N+1}(x_0), f(z)) \leq \gamma\varrho(f^N(x_0), z)$$

which contradicts

$$\gamma\varrho(f^N(x_0), z)) \leq \varrho(f^N(x_0), z) \leq \frac{1}{4}\varrho(z, f(z)) < \varrho(f^{N+1}(x_0), f(z))$$

$$\text{and}\quad \varrho(f^{N+1}(x_0), f(z)) \leq \beta(\varrho(f^N(x_0), f^{N+1}(x_0)) + \varrho(z, f(z))).$$

Contradicting

$$
\begin{aligned}
&\because\quad \beta(\varrho(f^N(x_0), f^{N+1}(x_0)) + \varrho(z, f(z)))\\
&< \frac{3}{5}(\varrho(f^N(x_0), z) + \varrho(z, f^{N+1}(x_0)) + \varrho(z, f(z)))\\
&< \frac{18}{20}\varrho(z, f(z))\\
&< \varrho(f^{N+1}(x_0), f(z)).
\end{aligned}
$$

Also

$$\varrho(f^{N+1}(x_0), f(z)) \leq \alpha(\varrho(f^n(x_0), f(z)) + \varrho(f^{N+1}(x_0), z)).$$

Contradicting

$$\begin{aligned}
&\because \quad \alpha(\varrho(f^N(x_0), f(z)) + \varrho(z, f^{N+1}(x_0))) \\
&< \frac{2}{5}(\varrho(f^N(x_0), z) + \varrho(z, f(z)) + \varrho(f^{N+1}(x_0), z)) \\
&< \frac{3}{5}\varrho(z, f(z)) < \varrho(f^{N+1}(x_0), f(z)) \\
&\therefore \quad f(z) = z.
\end{aligned}$$

Now, we show that, fixed point z is unique.
Suppose this is not true,
i.e. $f(z') = z'$ some point $z' \in X$ and $z \neq z'$. Then,

$$\varrho(f(z), f(z')) = \varrho(z, z')$$

$$\varrho(f(z), f(z')) > \varrho(z, f(z)) + \varrho(z', f'(z)).$$

$$\text{and} \quad \varrho(f(z), f(z')) = \frac{1}{2}(\varrho(z, f'(z)) + \varrho(z', f(z))).$$

$\therefore$ All three conditions of the theorem is not satisfied at the points z and z'.

$\therefore$ z is unique.

This shows that self-map f has unique fixed point.

Remark :

(1) From $0 \leq \alpha < \beta < \gamma < 1$ with $\alpha \leq \frac{2}{5}$ and $\beta \leq \frac{3}{5}$ we have $0 < \gamma < 1$ then Theorem 4.1 (a) reduce to Banach (Theorem 3.1).

(2) If $\gamma \leq \frac{1}{2}$ then Theorem 4.1(b) reduce to Kannan (Theorem 3.2).

(3) If $\beta = \frac{1}{2}$ then Theorem 4.1(c) reduce to Chatterjee (Theorem 3.3).

5. Conclusion

The above result shows that in complete metric space the self map f has a unique fixed point.

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