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MENGER SPACES IN COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS

M. VIJAYAKUMAR[1] AND S. M. SUBHANI[2]

Abstract

In this present paper menger space in common fixed point theorem for compatible mappings our results generalizes and extends .

1. Introduction

Probabilistic metric spaces, which is a overview of metric spaces was introduced by Menger [7]. The premise of probabilistic metric spaces is of elementary in probabilistic functional analysis was performed by [15] and others under various contractive conditions are worked by [10, 11, 13, 14, 21].

A common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem by [3] and defined weak commutativity and proved common fixed point theorem for weakly commuting mappings proved by Sessa [18]. Further, [4] introduced the notion of compatibility, which is more general than that of weak commutativity,

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then different fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings, have been obtained by many authors. In 1998, introduced the notion of weak compatibility and showed that compatible maps are weakly compatible but the converse need not to be true. Finally by [5, 17] proved common fixed point theorems for weakly compatible mappings on a complete Menger spaces without using the condition of continuity.
Finally, replacing the condition of compatibility of type (P) by weak-compatibility in complete Menger space. By Pathak [9].

2. Preliminaries

Definition 2.1 : A real valued function f on the set of real numbers is called a distribution function if it is non-decreasing, left continuous with $\inf_{u\in R} f(u) = 0$ and $\sup_{u\in R} f(u) = 1$.
OR Heaviside function H is a distribution function defined by

$$H(u) = \begin{cases} 0, & u \leq 0 \\ 1, & u > 0 \end{cases}$$

Definition 2.2 : Let X be a non-empty set and let L denote the set of all distribution functions defined on X. An ordered pair (X, T) is called a probabilistic metric space where T is a mapping from $X \times X$ into L if for every pair $(x, y) \in X$ a distribution function $F(x, y)$ or $F_{x,y}$ assumed to satisfy the following conditions:

(1) $F_{x,y}(u) = H(u)$ if and only if $x = y$.

(2) $F_{x,y}(u) = F_{y,x}(u)$.

(3) $F_{x,y}(0) = 0$.

(4) If $F_{x,y}(u_1) = 1$ and $F_{y,z}(u_2) = 1$, then $F_{x,z}(u_1 + u_2) = 1$ for all x, y, z in X and $u_1, u_2 \geq 0$.

Every metric space (X, d) can be realized as a probabilistic metric space by taking $T : X \times X \to L$ defined by $F_{x,y}(u) = H(u - d(x, y))$ for all x, y in X. So probabilistic metric spaces provide a wider framework than that of the metric spaces and are better suited in many situations.

Definition 2.3 : A t-norm is a function $t : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions:

(T1) $t(a,1) = a, t(0,0) = 0$,

(T2) $t(a,b) = t(b,a)$,

(T3) $t(c,d) \geq t(a,b)$ for $c \geq a,\ d \geq b$,

(T4) $t(t(a,b),c) = t(a,t(b,c))$ for all $a,b,c \in [0,1]$.

Definition 2.4 : A Menger probabilistic metric space $(X, \Im, t)$ is an ordered triple, where t is a t-norm, and $(X, \Im)$ is a probabilistic metric space satisfying the following condition: $F_{x,z}(u_1 + u_2) \geq t(F_{x,y}(u_1), F_{y,z}(u_2))$ for all x, y, z in X and $u_1, u_2 \geq 0$.
Definition 2.5 : A sequence $\{x_n\}$ in $(X, \Im, t)$ is said to

(I) Converge to a point $x \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\epsilon, \lambda)$ such that $F_{xn,xm}(\epsilon) > 1 - \lambda$ for all $n \geq N(\epsilon, \lambda)$.

(II) Cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\epsilon, \lambda)$ such that $F_{xn,xm}(\epsilon) > 1 - \lambda$ for all $n, m \geq N(\epsilon, \lambda)$.

(III) Continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 2.6 (I) : A coincidence point (or simply coincidence) of two mappings is a point in their domain having the same image point under both mappings.
Formally, given two mappings $f, g : X \to Y$ we say that a point x in X is a coincidence point of f and g if $f(x) = g(x)$.
(II) A pair of mappings A and S is called a weakly compatible pair if they commute at a coincidence point.
Example 2.1 : Define the pair $A, S : [0,5] \to [0,5]$ by

$$A(x) = \begin{cases} x, & x \in [0,1] \\ 5, & x \in [1,5], \end{cases} \qquad S(x) = \begin{cases} 5 - x, & x \in [0,1] \\ 5, & x \in [1,5]. \end{cases}$$

Then for any $x \in [1,5]$, $ASx = SAx$, showing that A, S are weakly compatible maps on $[0,3]$.
Definition 2.7 : An PM-space (X, T) is said to be a simple space if and only if there exists a metric d on X and a distribution function G satisfying $G(0) = 0$, such that for

every x, y in X

$$F_{x,y}(u) = \begin{cases} G\left(\frac{u}{d(x,y)}\right), & x \neq y; \\ & \\ H(u) & x = y \end{cases} \quad \text{for all} \quad x, y \in X.$$

Furthermore, we say that (X, T) is the simple space generated by the metric space (X, d) and the distribution function G.

Remark [15] : A simple space is a Menger space under any choice of T satisfying (T1), (T2), (T3),(T4).

Lemma 2.11 ([20, 16]) : Let $\{x_n\}$ be a sequence in a Menger space $(X, \Im, t)$, where t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$. If there exists a constant $k \in (0, 1)$ such that, $F_{xn,xn+1}(kx) \geq F_{xn-1,xn}(x)$ for all $x > 0$ and $n \in N$, then $\{x_n\}$ is Cauchy sequence in X.

Now we prove a common fixed point theorem for four weakly compatible maps on a complete Menger space.

3. Main Theorem

Theorem 3.1 : Let A, B, S and T be self mappings on a complete Menger space (X, F, t) where $t(x, y) = \min(x, y)$ for all $x, y \in [0, 1]$, satisfying the following conditions:

3.1(i) $A(X), B(X)$ are closed sets of X and $A(X) \subset T(X)$, $B(X) \subset S(X)$,

3.1(ii) the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible,

3.1(iii) $F(M_{Ax,By}, (kt)M_{Sx,Ty}, (t), M_{Ax,Sx}, (t), M_{By,Ty}, (kt), M_{Ax,Ty}, (t)) \geq 0$.

If the pair $\{A, S\}$ is reciprocal continuous, semi-compatible maps. Then A, B, S and T have a unique common fixed point in X.

bf Proof : Let $x_0 \in X$, be any arbitrary point. From 3.1(i), there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1 = y_0$ and $Bx_1 = Sx_2 = y_1$. Inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, \cdots$. Putting $x = x_{2n}, y = x_{2n+1}$ for $t > 0$ in (3); we get

$$F\{M_{Ax_{2n},Bx_{2n+1}}(kt),\ M_{Sx_{2n},Tx_{2n+1}}(t),\ \ M_{Ax_{2n},Sx_{2n}}(t),\ \ M_{Bx_{2n+1},Tx_{2n+1}}(kt)\} \geq 0.$$

That is,

$$F\{M_{y_{2n}y_{2n+1}}(kt),\ M_{y_{2n-1},y_{2n}}(t),\ \ M_{y_{2n},y_{2n-1}}(t),\ \ M_{y_{2n+1},y_{2n}}(kt)\} \geq 0.$$

Using (i), we get

$$M_{y_{2n},y_{2n+1}}(kt), \quad M_{y_{2n},y_{2n+1}}(t) \geq 0.$$

Again we put $x = x_{2n+2}$ and $y = x_{2n+3}$; we have

$$F\{M_{Ax_{2n+2},Bx_{2n+1}}(kt),\ M_{Sx_{2n+2},Tx_{2n+3}}(t),\ M_{Ax_{2n+2},Sx_{2n+2}}(t),\ M_{Bx_{2n+3},Tx_{2n+3}}(kt)\} \geq 0.$$

That is

$$F\{M_{y_{2n+3}y_{2n+2}}(kt),\ M_{y_{2n+2},y_{2n+1}}(t), \quad M_{y_{2n+2},y_{2n+1}}(t), \quad M_{y_{2n+3},y_{2n+2}}(kt)\} \geq 0.$$

Using (i); we get

$$M_{y_{2n+3},y_{2n+2}}(kt), \quad M_{y_{2n+2},y_{2n+1}}(t) \geq 0.$$

Thus for any n and t, we have

$$M_{y_n,y_{n+1}}(kt) \ \ M_{y_{n-1}y_n}(t) \geq 0.$$

Hence by Lemma 2.11 $\{y_n\}$ is a Cauchy sequence in X. Since X is complete therefore $\{y_n\} \to z$ in X and its subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\}, \{Sx_{2n}\}$ also converges to z. (A, S) is reciprocally continuous mapping then we have,

$$\lim_{n\to\infty} ASx_{2n} = Az, \qquad \lim_{n\to\infty} SAx_{2n} = Sz$$

and semi-compatibility of (A, S) gives;

$$\lim_{n\to\infty} ASx_{2n} = Sz.$$

Hence $Az = Sz$.

Step 1 : By putting $x = z$, $y = x_{2n+1}$ in (3), we get

$$F\{M_{Az,Bx_{2n+1}}(kt), \quad M_{Sx,Tx_{2n+1}}(t) \ \ M_{Az,Sz}(t), \quad M_{Bx_{2n+1},Tx_{2n+1}}(kt)\} \geq 0.$$

Letting $n \to \infty$, we get

$$F\{M_{Az,z}(kt), \quad M_{Az,z}(t), \quad M_{Az,Az}(t), \quad M_{z,z}(t)\} \geq 0.$$

As F is non-decreasing in the first argument, we have

$$F\{M_{Az,z}(t),\ M_{Az,z}(t), 1, 1\} \geq 0.$$

That is

$$M_{Az,z}(t) \geq 1.$$

Therefore $Az = z = Sz$.

Step 2 : As $A(x) \subseteq T(x)$, there exists $u \in X$ such that $z = Az = Tu$, putting $x = x_{2n}$ and $y = u$ in (3), we get

$$F\{M_{Ax_{2n},Bu}(kt), \quad M_{Sx_{2n},Tu}(t), \quad M_{Ax_{2n},Sx_{2n}}(t), \quad M_{Bu,Tu}(kt)\} \geq 0.$$

Letting $n \to \infty$; we get

$$F\{M_{z,Bu}(kt), \quad M_{z,z}(t), \quad M_{z,z}(t),, \quad M_{Bu,z}(kt)\} \geq 0.$$

As F is non-decreasing in the first argument, we have

$$F\{M_{z,Bu}(t), 1, 1, \ M_{Bu,z}(t), 1\} \geq 0.$$

That is

$$M_{z,Bu}(t) \geq 1.$$

Therefore, $z = Bu = Tu$.

i.e., $Tu = TBu \Rightarrow z = Tz$.

Putting $x = z$ and $y = z$ in (3), we get

$$F\{M_{Az,Bz}(kt), \quad M_{Sz,Tz}(t), \quad M_{Az,Sz}(t), \quad M_{Bz,Tz}(kt)\} \geq 0.$$

We get

$$F\{M_{z,Bz}(kt), \quad M_{z,z}(t), \quad M_{z,z}(t), \quad M_{Bz,z}(kt)\} \geq 0.$$

As F is non-decreasing in the first argument, we have

$$F\{M_{z,Bz}(kt), \ 1, 1, \ M_{Bz,z}(t), `\} \geq 0.$$

That is

$$M_{z,Bu}(t) \geq 1.$$

Therefore $z = Bz = Tz$. Hence $Az = Bz = Sz = tz = z$.

Uniqueness : Let w be another fixed point of A, B, S and t. Therefore putting $x = z$ and $y = w$ in (3), we have

$$F\{M_{Az,Bw}(kt), \ M_{Sz,Tw}(t), \quad M_{Az,Sz}(t), \quad M_{Bz,Tz}(kt)\} \geq 0$$

$$F\{M_{z,w}(kt),\quad M_{z,w}(t),` M_{z,z}(kt),` M_{z,z}(t)\} \geq 0.$$

As F is non-decreasing in the first argument, we have

$$F\{M_{z,w}(t),\quad M_{z,w}(t), 1, 1\} \geq 0.$$

i.e., $z = w$. Hence z is a unique fixed point in X.

Corolloary 3.2 : Let A, S and T be self mappings of a complete fuzzy metric space $(X, M, *)$ with t-norm defined by $a * b = \min\{a, b\}$, satisfying :

3.2(i) $A(X) \subseteq S(X) \cap T(X)$;

3.2(ii) A is T absorbing;

3.2(iii) for some $F \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$F\{M_{Ax,Ay}(kt),\quad M_{Sx,Ty}(t),\quad M_{Ax,Sx}(t),\quad M_{Ay,Ty}(kt)\} \geq 0.$$

If the pair $\{A, S\}$ is reciprocal continuous, semi-compatible maps. Then A, S and T have a unique common fixed point in X.

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