International J. of Pure & Engg. Mathematics (IJPEM)
ISSN 2348-3881, Vol. 4 No. II (August, 2016), pp. 43-46

# APPROXIMATION OF ALTERNATING SERIES

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**Abstract**

In this paper we shall give an approximation of alternating series using correction function. The introduction of correction function certainly improves the sum of the series and gives a better approximation to it.

## 1. Introduction

The sum of an alternating series can be approximated by its sequence of partial sums. Then the error obtained is the remainder term of the series. The absolute value of the remainder term plays a vital role in the approximation of alternating series.

**Definition 1** : An **alternating series** is a series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ where the terms $a_n > 0$.

**Definition 2** : The **remainder term** for an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is the sum of the series after $n$ terms. It is denoted by $R_n$.

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Key Words : *Alternating series, Remainder term, Correction function, Forward difference, Sequence of partial sums.*

i.e. $R_n = \sum\limits_{k=n+1}^{\infty} (-1)^{k-1} a_k$.
If $S$ denote the sum of the series and $S_n$ denote the sequence of partial sums of the series, then $R_n == S - S_n$.
**Definition 3** : The **correction function** to an alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ is denoted by $G_n$ and it is defined as the absolute value of the remainder term.
i.e. If $R_n$ denotes the remainder term of the series, then

$R_n = (-1)^n G_n$ where $G_n$ is the correction function.

i.e. $G_n = \sum\limits_{k=1}^{\infty} (-1)^{k-1} a_{n+k}$.
If $\{a_n\}$ is monotonically decreasing, then $|S - S_n| = G_n$.
**Definition 4** : Let $S$ denote the sum and $S_n$ denote the sequence of partial sums of the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$. Suppose that $\{a_n\}$ is monotonically decreasing.
Then we define the **forward differences of the terms** as

$$\Delta a_n = a_n - a_{n+1}$$

In general,

$$\Delta^k a_n = \Delta^{k-1} a_n - \Delta^{k-1} a_{n+1} \quad \text{for} \quad k > 1.$$

Then the correction function

$$G_n = \Delta a_{n+1} + \Delta a_{n+3} + \Delta a_{n+5} + \Delta a_{n+7} + \cdots$$

$$\text{i.e.} \quad |S - S_n| = \Delta a_{n+1} + \Delta a_{n+3} + \Delta a_{n+5} + \Delta a_{n+7} + \cdots$$

**Theorem 1** : Let $S_n = \sum\limits_{i=1}^{n} (-1)^{i-1} a_i$ be the $n^{\text{th}}$ partial sum of an alternating series and let $S = n\ n \overset{Lt}{\rightarrow} \infty\ S_n$. Suppose that $0 < a_{n+1} < a_n$ for all $n$ and $\lim\limits_{n\to\infty} a_n = 0$. Then $G_n = |S - S_n| < a_{n+1}$.
**Proof** : Let

$$\begin{aligned} S &= \sum_{n=1}^{\infty} (-)^{n-1} a_n \\ &= S_n + (-1)^n a_{n+1} + (-1)^{n+1} a_{n+2} + (-1)^{n+2} a_{n+3} + \cdots \\ &= S_n + (-1)^n \{a_{n+1} - a_{n+2} + a_{n+3} - \cdots\} \end{aligned}$$

$$S - S_n = (-1)^n \{a_{n+1} - a_{n+2} + a_{n+3} - \cdots\}$$

$$
\begin{aligned}
|S - S_n| &= |\{a_{n+1} - a_{n+2} + a_{n+3} - \cdots\}| \\
&= +\{a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) \cdots\}| \\
&< |a_{n+1}| \quad \text{(since the terms } a_n \text{ monotonically decreases with } n) \\
&= a_{n+1} \quad (\text{since } a_{n+1} > 0) \\
|S - S_n| &< a_{n+1}
\end{aligned}
$$

i.e. $G_n < a_{n+1}$.

Hence the proof.

**Theorem 2** : For a convergent alternating series $\sum_{n=1}^{\infty}(-)^{n-1}a_n$ if the terms $a_n$ are monotonically decreasing and if $a_n \leq 2\epsilon$, then $G_n < \epsilon$.

**Proof** : Let $S$ denote the sum of the series $\sum_{n=1}^{\infty}(-)^{n-1}a_n$ whose sequence of partial sums is $S_n$. Then the correction function after $n$ terms is $G_n = |S - S_n|$.

Let $\Delta a_n = a_n - a_{n+1}$.

Then by our assumption, $\Delta a_{n+1} < \Delta a_n$. Also we have

$$
\begin{aligned}
G_n &= \{a_{n+1} - a_{n+2} + a_{n+3} - \cdots\} \\
&= \{(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + (a_{n+5} - a - n + 6) \cdots\} \\
&= \{\Delta a_{n+1} + \Delta a_{n+3} + \Delta a_{n+5} + \Delta a_{n+7} \cdots\}
\end{aligned}
$$

so that $G_{n-1} = \{\Delta a_n + \Delta a_{n+2} + \Delta a_{n+4} + \Delta a_{n+6} \cdots\}$.

Since $(\Delta a_n)$ is monotonically decreasing, we have $G_n < G_{n-1}$.

Also we have $a_n = G_n + G_{n-1}$.

If $a_n \leq 2\epsilon$, then $G_n + G_{n-1} \leq 2\epsilon$.

Since $G_n < G_{n-1}$, it follows that $G_n < \epsilon$.

Since $G_{n+k} < G_n$, for all $k \in N$, it follows that $G_{n+k} < \epsilon$ for all $k$.

i.e. $|S_{n+k} - S| < \epsilon$ for all $k \in N$.

i.e. $G_{n+k} < \epsilon$ for all $k \in N$.

i.e. All partial sums following $S_n$ will lie in the limit of accuracy.

Hence the proof.

## References

[1] Knopp Konrad, Theory and Application of Infinite Series, Blackie and Son Limited (London and Glasgow).

[2] Hardy G. H., A Course of Pure Mathematics, Cambridge at the University Press, (Third Edition), (1963).

[3] Knopp K., Infinite Sequences and Series, Dover, (1956).

[4] Calabrese Phillip, A note on alternating series, The American Mathematical Monthly, 69(3) (May, 1962), 215-17.

[5] Morley R. K., The remainder in computing by series, The American Mathematical Monthly, 57(80 (October 1950), 550-51.