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THE THIRD HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY CARLSON-SHAFFER OPERATOR

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Abstract

The objective of this paper is to be obtain upper bond to the Third Hankel determinant denoted by H_3 (1) for certain subclass of univalent function by using Carlson-Shaffer operator.

1. Introduction

Let A be the class of analytic function in the open unit disk.

$$E = \{z : z\epsilon E, |z| < 1\} \tag{1.1}$$

Key Words and Phrases : *Analytic function, Univalent function, Coefficient bounds, Third Hankel determinant, Carlson-Shaffer operator.*

We denote A_0 the subclass A consisting of normalized function of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (Z \epsilon E) \tag{1.2}$$

For the functions f & g in A given by the series expansion,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n \qquad (Z \epsilon E) \tag{1.3}$$

The Hadamard product $f * g$ is defined by,

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \qquad (Z \epsilon E) \tag{1.4}$$

The function $f * g \ \epsilon$ A we recall that the Carlson-Shaffer operator[4],

$$\mathfrak{L}(\alpha, \beta) = A_0 \to A_0 \quad (\alpha \epsilon \mathbf{C} \quad \beta \epsilon \mathbf{C}/\bar{Z}_0; \quad \bar{Z}_0 = \{0, -1, -2, \ldots\}) \tag{1.5}$$

is defined by,

$$\begin{aligned} \mathfrak{L}(\alpha, \beta) f(z) &= \phi(\alpha, \beta; z) * f(z) \qquad \{z \ \epsilon \ E, \ f \ \epsilon \ A\} \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha)_{n-1}}{(\beta)_{n-1}} a_n z^n \end{aligned} \tag{1.6}$$

where,

$$\phi(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} z^{k+1} \qquad (z \ \epsilon \ E) \tag{1.7}$$

and $(\lambda)_k$ is the Pochhammer Symbol (or Shifted Factorial) defined in terms of Gamma Function by,

$$(\lambda)_k = \frac{\Gamma \lambda + k}{\Gamma \lambda} = \begin{cases} 1 & k = 0 \\ \lambda(\lambda+1)(\lambda+2)\ldots & (k \epsilon N = \{1, 2, \ldots\}) \end{cases} \tag{1.8}$$

For the complex sequence a_n, a_{n+1}, a_{n+2}, ... The Hankel matrix after Herman Hankel (1839-1873), is the infinite matrix whose $(i,j)^{th}$ entry a_{ij} defined by,

$$a_{i,j} = a_{i-i,j+1} \qquad (i, j \epsilon N \backslash \{1\}) \tag{1.9}$$

The q^{th} Hankel matrix is by definition the following $q \times q$ Square Sub Matrix. The determinant of the q^{th} Hankel usually denoted by,

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & & & \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{1.10}$$

In particular cases, q = 2, n = 1, a = 1, & q = 2, n = 2, the Hankel determinant simplifies respective to,

$$H_2(1) = |a_3 - a_2^2| \tag{1.11}$$

$$H_2(1) = |a_2 a_4 - a_3^2| \tag{1.12}$$

In this paper, we consider the Hankel determinant in the case q = 3 & n = 1.

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \tag{1.13}$$

$H_3 1$ is called third hankel determinant.

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2| \tag{1.14}$$

Second Hankel Determinant for various classes has been extensively studied by various authors including Singh [10][11], Mehrok & Singh [9] & Janteng et al [4][5][6].
But the Third Hankel determinant has been studied by some of the researchers including Babalola [2], Shanmugam et al [13] and Gagandeep Singh et al [1]. Motivated by the result obtained by Gagandeep Singh et al [1] & Babalola [2].
We obtain upper bonds by using Carlson-Shaffer operator to the fractional $|a_2 a_3 - a_4|$ & hence $H_3(1)$ for function given by (1.1), when it belongs to the class $\Re_{\alpha,\beta}(\lambda)$ defined as follows,
Definition 1.1 : The function f ϵ A_0 is said to be in the class $\Re_{\alpha,\beta}(\lambda)$, $0 \leq \lambda < 1$.

$$\Re\Big\{(1-\lambda)\frac{\mathfrak{L}(\alpha\beta)f(z)}{z} + \lambda[\mathfrak{L}(\alpha\beta)f(z)]'\Big\} > 0 \qquad (Z\epsilon E) \tag{1.15}$$

If $\alpha = \beta$=1, the class R(λ) is the class studied by Murugusundramurthi & Magesh [11] & if $\lambda = 1$, R(1) = R the class of functions whose derivative has a real part introduce & studied by McGregor [9]. In the present paper we obtained an upper bond for functional $H_3(1)$ for the function in the class $R_{\alpha,\beta}(\lambda)$.

2. Preliminary Results

Lemma 2.1 : Let P be the family of all functions p analytic in E for which,

$$\begin{gathered} \Re\{p(z)\} > 0 \\ p(z) = 1 + p_1 z + p_2 z^2 + \dots \qquad z\epsilon E \end{gathered} \tag{2.1}$$

Lemma 2.2 : If p ϵ P then,

$$|P_k| \leq 2 \qquad (k = 1, 2, 3, \ldots) \tag{2.2}$$

Lemma 2.3 : If p ϵ P then,

$$\begin{aligned} 2P_2 &= P_1^2 + (4 - P_1^2)x \\ 4P_3 &= P_1^3 + 2P_1(4 - P_1^2)x - P_1(4 - P_1^2)x^2 + 2(4 - P_1^2)(1 - |x|^2)z \end{aligned} \tag{2.3}$$

for some x & z satisfying, $|x| \leq 1$, $|z| \leq 1$ & P ϵ [0,z].

Lemma 2.4 [11] : If f(z) ϵ $\Re_{\alpha,\beta}(\lambda)$ then,

$$|a_2a_4 - a_3^2| \leq \frac{4\beta^2(1+\beta)(\beta+2)}{\alpha^2(\alpha+1)(\alpha+2)(1+2\lambda)^2} \tag{2.4}$$

If $\alpha = 1$, $\beta = 1$, we get,

$$|a_2a_4 - a_3^2| \leq \frac{4}{(1+2\lambda)^2} \tag{2.5}$$

3. Main Results

Theorem 3.1 : Let f ϵ $\Re_{\alpha,\beta}(\lambda)$ then we have the best possible bound for

$$|a_2a_3 - a_4| \leq \begin{cases} \frac{2}{1+3\lambda} & If \quad \alpha = \beta = 1 \\ \frac{2\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)(1+3\lambda)} & (0 \leq \lambda < 1, \alpha\epsilon\mathbf{C}, \beta\epsilon\mathbf{C}/Z_{\bar{0}}) \end{cases} \tag{3.1}$$

Proof : Let f ϵ $\Re_{\alpha,\beta}(\lambda)$ $(0 \leq \lambda < 1)$ then there exist p ϵ P such that,

$$(1-\lambda)\frac{\mathfrak{L}(\alpha,\beta)f(z)}{z} + \lambda\Big[\mathfrak{L}(\alpha,\beta)f(z)\Big]' = P(z) \tag{3.2}$$

$$(1-\lambda)\Big[\frac{z + \sum_{n=2}^{\infty}\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}a_nz^n}{z}\Big] + \lambda\Big[1 + \sum_{n=2}^{\infty}\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}a_nz^{n-1}\Big] = P(z) \tag{3.3}$$

$$(1-\lambda)\Big[1+\sum_{n=2}^{\infty}\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}a_nz^{n-1}\Big] + \lambda\Big[1+\sum_{n=2}^{\infty}\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}a_nz^{n-1}\Big] = 1 + P_1z + P_2z^2 + \ldots \tag{3.4}$$

Equating coefficients in (3.3) yields,

$$(1-\lambda)\Big[\frac{\alpha}{\beta}a_2\Big] + (\lambda)\Big[\frac{2\alpha}{\beta}a_2\Big] = P_1 \tag{3.5}$$

$$\frac{\alpha}{\beta}a_2[1 - \lambda + 2\lambda] = P_1$$

$$a_2 = \frac{P_1}{(1+\lambda)\frac{\alpha}{\beta}} \tag{3.6}$$

$$a_3 = \frac{P_2}{(1+2\lambda)\frac{(\alpha)_2}{(\beta)_2}} = \frac{P_2}{(1+2\lambda)\frac{\alpha(\alpha+1)}{\beta(\beta+1)}} \tag{3.7}$$

$$a_4 = \frac{P_3}{(1+3\lambda)\frac{(\alpha)_3}{(\beta)_3}} = \frac{P_3}{(1+3\lambda)\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}} \tag{3.8}$$

$$a_5 = \frac{P_4}{(1+4\lambda)\frac{(\alpha)_4}{(\beta)_4}} = \frac{P_4}{(1+4\lambda)\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\beta(\beta+1)(\beta+2)(\beta+3)}} \tag{3.9}$$

$$\begin{aligned} |a_2a_3 - a_4| &= \left|\frac{P_1P_2}{(1+\lambda)(1+2\lambda)\frac{\alpha}{\beta}\times\frac{\alpha(\alpha+1)}{\beta(\beta+1)}} - \frac{P_3}{(1+3\lambda)\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}}\right| \\ &= \left|\frac{\beta^2(\beta+1)P_1P_2}{(1+\lambda)(1+2\lambda)\alpha^2(\alpha+1)} - \frac{P_3\beta(\beta+1)(\beta+2)}{(1+3\lambda)\alpha(\alpha+1)(\alpha+2)}\right| \end{aligned} \tag{3.10}$$

By using Lemma,

$$2P_2 = \frac{P_1^2 + x(4-P_1^2)}{2}$$

$$4P_3 = P_1^3 + 2P_1(4-P_1^2)x - P_1(4-P_1^2)x^2 + 2(4-P_1^2)(1-|x|^2)z$$

$$|a_2a_3 - a4| = \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\left|\frac{\beta P_1P_2}{(1+\lambda)(1+2\lambda)\alpha} - \frac{\beta+zP_3}{(1+3\lambda)(\alpha+2)}\right|$$

$$= \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\left|\frac{\beta P_1[\frac{P_1^2+x(4-P_1^2}{2}]}{(1+\alpha)(1+2\alpha)\alpha} - \frac{\beta+2}{(1+3\lambda)(\alpha+2)}\right.$$

$$\left.\Big(\frac{P_1^3 + 2P_1(4-P_1^2)x - P_1(4-P_1^2)x^2 + 2(4-P_1^2)(1-|x|^2)z}{4}\Big)\right|$$

$$= \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\left|\frac{\beta P_1^3 + \beta p_1x(4-P_1^2)}{2(1+\lambda)(1+2\lambda)\alpha} - \frac{(\beta+2)}{4(1+3\lambda)(\alpha+2)}\right.$$

$$\left.[P_1^3 + 2P_1(4-P_1^2)x - P_1(4-P_1^2)x^2 + 2(4-P_1^2)(1-|x|^2)z]\right|$$

$$= \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\left|\frac{\beta P_1^3 + \beta p_1x(4-P_1^2)}{2(1+\lambda)(1+2\lambda)\alpha} - \frac{(\beta+2)}{4(1+3\lambda)(\alpha+2)} - \right.$$

$$\frac{P_1(4-P_1^2x(\beta+2)}{2(1+3\lambda)(\alpha+2)} + \frac{(\beta+2)P_1(4-P_1^2x^2}{4(1+3\lambda)(\alpha+2)} -$$

$$\left.\frac{2(\beta+2)P_1(4-P_1^2(1-|x|^2)z}{4(1+3\lambda)(\alpha+2)}\right|$$

Since $|P| = |P_1| \le 2$. We may assume that p ϵ [0,2]. Then using triangle inequality that $|z| \le 1$ with $\rho = |x|$ we obtain,

$$\begin{aligned} &\le \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\Big[\frac{\beta P^3 + \beta P\rho(4-P^2)}{2(1+\lambda)(1+2\lambda)\alpha} + \frac{(\beta+2)P^3}{4(1+3\lambda)(\alpha+2)} + \frac{P(4-P^2\rho(\beta+2)}{2(1+3\lambda)(\alpha+2)} \\ &+ \frac{(\beta+2)P(4-P^2)\rho^2}{4(1+3\lambda)(\alpha+2)} + \frac{2(\beta+2)(4-P^2)(1-\rho^2)}{4(1+3\lambda)(\alpha+2)}\Big] = F(\rho) \end{aligned} \tag{3.11}$$

Then,

$$F'(\rho) = \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\Big[\frac{\beta P(4-P^2)}{2(1+\lambda)(1+2\lambda)\alpha} + \frac{P(4-P^2)(\beta+2)}{2(1+3\lambda)(\alpha+2)} + \frac{2\rho(\beta+2)(4-P^2)}{4(1+3\lambda)(\alpha+2)} + \frac{(\beta+2)2\rho(4-P^2)}{4(1+3\lambda)(\alpha+2)}\Big]$$
$$= \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\Big[\frac{\beta P(4-P^2)}{2(1+\lambda)(1+2\lambda)\alpha} + \frac{P(4-P^2)(\beta+2)}{2(1+3\lambda)(\alpha+2)} \frac{2(\beta+2)(4-P^2)\rho}{4(1+3\lambda)(\alpha+2)}\Big] \tag{3.12}$$

F(ρ) is decreasing function on [0,2] so that F(ρ) $\leq$ F(0).

$$|a_2a_3 - a_4| \leq \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\Big[\frac{\beta P(4-P^2)}{2(1+\lambda)(1+2\lambda)\alpha} + \frac{P(4-P^2)(\beta+2)}{2(1+3\lambda)(\alpha+2)} + \frac{(\beta+2)(4-P^2)}{2(1+3\lambda)(\alpha+2)}\Big] = G(P) \tag{3.13}$$

$$G(P) = \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\Big[\frac{\beta P(4-P^2)}{2(1+\lambda)(1+2\lambda)\alpha} + \frac{P(4-P^2)(\beta+2)}{2(1+3\lambda)(\alpha+2)} + \frac{(\beta+2)(4-P^2)}{2(1+3\lambda)(\alpha+2)}\Big] \tag{3.14}$$

Obviously G(P) is increasing function in [0,2], therefore G(P) $\leq$ G(0).

$$|a_2a_3 - a_4| \leq \frac{2\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)(1+3\lambda)} \tag{3.15}$$

If $\alpha = 1$, $\beta = 1$, then we get $|a_2a_3 - a_4| \leq \frac{2}{(1+3\lambda)}$ is result of Gagandeep Singh et al [1]. □

Theorem 3.2 : f ϵ $\Re_{\alpha,\beta}(\lambda)$, then

$$|a_3 - a_2|^2 \leq \frac{4\beta(\beta+1)}{(2\alpha)(\alpha+1)(1+2\lambda)}$$

Proof : Since f ϵ $R_{\alpha,\beta}(\lambda)$

$$|a_3 - a_2^2| = \left|\frac{P_2\beta(\beta+1)}{\alpha(\alpha+1)(1+2\lambda)} - \frac{P_1^2\beta^2}{\alpha^2(1+\lambda)^2}\right|$$
$$= \left|\frac{\beta(\beta+1)P_2}{\alpha(\alpha+1)(1+2\lambda)} - \frac{P_1^2\beta^2}{\alpha^2(1+\lambda)^2}\right|$$
$$By \quad using \quad Lemma,$$
$$= \left|\frac{\beta(\beta+1)\left[\frac{P_1^2+x(4-P_1^2)}{2}\right]}{\alpha(\alpha+1)(1+2\lambda)} - \frac{P_1^2\beta^2}{\alpha^2(1+\lambda)^2}\right|$$
$$= \left|\frac{\beta(\beta+1)[P_1^2+x(4-P_1^2)]}{2\alpha(\alpha+1)(1+2\lambda)} - \frac{P_1^2\beta^2}{\alpha^2(1+\lambda)^2}\right| \tag{3.16}$$

Since $|P| = |P_1| \leq$ z, we assume that P ϵ [0,2], $|z| \leq 1$.

Using triangle inequality $\rho = |x|$.

$$\leq \frac{\beta(\beta+1)[P^2 + \rho(4-P^2)]}{2\alpha(\alpha+1)(1+2\lambda)} + \frac{P^2\beta^2}{\alpha^2(1+\lambda)^2} = F(\rho) \tag{3.17}$$

$$F'(\rho) = \frac{\beta(\beta+1)(4-P^2)}{2\alpha(\alpha+1)(1+2\lambda)} \tag{3.18}$$

$$F(1) = G(P) = \frac{\beta(\beta+1)[P^2+(4-P^2)]}{2\alpha(\alpha+1)(1+2\lambda)} + \frac{P^2\beta^2}{\alpha^2(1+\lambda)^2} \tag{3.19}$$

Obviously G(P) is increasing in [0,2], therefore G(P) $\leq$ G(0).

$$|a_3 - a_2^2| \leq \frac{4\beta[\beta+1]}{2\alpha(\alpha+1)(1+2\lambda)} \tag{3.20}$$

By using lemma 2.1 & theorem 3.1, 3.2 we get,

$$\begin{aligned} &|H_3(1)| = |a_3||a_2a_4a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \\ &= \left|\tfrac{2\beta(\beta+1)}{(1+\alpha)(1+2\lambda)\alpha}\right|\left|\tfrac{4\beta^2(\beta+1)(\beta+2)}{\alpha^2(\alpha+1)(\alpha+2)(1+2\lambda)^2}\right| + \\ &\left|\tfrac{2\beta(\beta+1)(\beta+2)}{\alpha(1+3\lambda)(\alpha+1)(\alpha+2)}\right|\left|\tfrac{2\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)(1+3\lambda)}\right| \\ &+\left|\tfrac{2\beta(\beta+1)(\beta+2)(\beta+3)}{(1+4\lambda)\alpha(\alpha+1)(\alpha+2)(\alpha+3)}\right|\left|\tfrac{4\beta(\beta+1)}{2\alpha(1+\alpha)(1+2\lambda)}\right| \end{aligned} \tag{3.21}$$

$$\begin{aligned} |H_3(1)| \leq \Big[&\tfrac{8\beta^3(\beta+1)^2(\beta+2)}{\alpha^3(\alpha+1)^2(\alpha+2)(1+2\lambda)^3} + \tfrac{4\beta^2(\beta+1)^2(\beta+2)^2}{\alpha^2(\alpha+1)^2(\alpha+2)^2(1+3\lambda)^2} + \\ &\tfrac{8\beta^2(\beta+1)^2(\beta+2)(\beta+3)}{2\alpha^2(1+\alpha)^2(1+2\lambda)(1+4\lambda)(\alpha+2)(\alpha+3)}\Big] \end{aligned} \tag{3.22}$$

□

Corollary 3.1 : If α =1, $\beta = 1$, $\lambda = 0$,

$|H_3(1)| \leq 16$, results coincides with that of Gagandeep Singh et al [1].

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