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SOME FIXED POINT THEOREMS IN COMPLETE METRIC SPACE

R. KRISHNAKUMAR[1] AND T. MANI[2]
[1] Department of Mathematics,
Urumu Dhanalakshmi College, Kattur, Tiruchirappalli-620019, India
[2] Department of Mathematics,
Trichy Engineering College, Konalai, Tiruchirappalli-621132, India

Abstract

In this paper, some fixed point theorems were proved in complete metric spaces which are the generalization of some existing results in the literature.

1. Introduction

Let T be a mapping on a complete metric space (X, d). Finding fixed point of T has a contraction mapping in non convex metric spaces is proved by P. V. Subrahmanyam [4]. There are so many theorems which proved the existence of unique fixed point of T, such as Banach's [8], Ciric's [3], Kannan's [2], Kirk's [11] and Meir and Keeler's [1].
In this paper, the generalization of some fixed point theorems were proved in the framework of metric spaces.

Key Words : *Metric space, Complete metric space, Contraction mapping, Fixed point, Unique fixed point.*

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Definition 1.1 : Let X be a non empty set, a function $d : X \times X \to R$ is called a metric on X if it satisfies the following conditions with

i. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$, $\forall x, y \in X$,

ii. $d(x, y) = d(y, x)$, $\forall x, y \in X$,

iii. $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$,

Then (X, d) is called a metric space.

Example 1 : Let $X = \mathbb{R}$ and $d : X \times X \to R$ such that

$$d(x, y) = |x - y|.$$

Then (X, d) is a metric space.

Definition 1.2 : Let (X, d) be a metric space and $\{x_n\}_{n\geq 0}$ be a sequence in X. Then $\{x_n\}_{n\geq 0}$ converges to x in X whenever for every $\epsilon > 0$ there is a natural number $n \in N$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 1.3 : Let (X, d) be a metric space and $\{x_n\}_{n\geq 0}$ be a sequence in X. $\{x_n\}_{n\geq 0}$ is a Cauchy sequence whenever for every $\epsilon > 0$ there is a natural number $n \in \mathbb{N}$, such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 1.4 : Let (X, d) be a metric space, if every Cauchy sequence is convergent in X, then X is called a complete metric space.

2. Main Result

Theorem 2.1 : Let (X, d) be a complete metric space. Suppose the mapping $T : X \to X$ satisfies the following conditions:

$$d(Tx, Ty) \leq \Big(\frac{d(x, Tx) + d(y, Ty)}{d(x, Tx) + d(y, Ty) + k}\Big) d(x, y) \tag{1}$$

for all $x, y \in X$, where $k \geq 1$. Then

(i) T has unique fixed point in X.

(ii) $T^n x^*$ converges to a fixed point, for all $x^* \in X$.

Proof : (i) Let $x_0 \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that $x_{n+1}=Tx_n$.

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \Big(\frac{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1}) + k}\Big) d(x_n, x_{n-1}) \\ &\leq \Big(\frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + k}\Big) d(x_n, x_{n-1}) \end{aligned}$$

Take

$$\beta_n = \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + k},$$

we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \beta_n d(x_n, x_{n-1}) \\ &\leq (\beta_n \beta_{n-1}) d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq (\beta_n \beta_{n-1} \cdots \beta_1) d(x_1, x_0). \end{aligned}$$

Observe that (β_n) is non increasing, with positive terms. So, $\beta_1 ... \beta_n \leq \beta_1^n$ and $\beta_1^n \to 0$. It follows that

$$\lim_{n\to\infty} (\beta_1 \beta_2 \cdots \beta_n) = 0.$$

Thus, it is verified that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$$

Now for all $m, n \in \mathbb{N}$ and $m > n$ we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq [(\beta_n \beta_{n-1} \cdots \beta_1) + (\beta_{n+1} \beta_n \cdots \beta_1) + \cdots + (\beta_{m-1} \beta_{m-2} \cdots \beta_1)] d(x_1, x_0) \\ d(x_m, x_n) &= \sum_{k=n}^{m-1} (\beta_k \beta_{k-1} \cdots \beta_1) d(x_1, x_0) \end{aligned}$$

Where, $a_k = (\beta_k \beta_{k-1} \cdots \beta_1)$.
Now, $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, $\sum_{k=1}^{\infty} a_k$ is finite and $\sum_{k=n}^{m-1} (\beta_k \beta_{k-1} \cdots \beta_1) \to 0,$ as $m, n \to \infty$.
Hence $\{a_k\}$ is convergent by D'Alembert's ratio test.

Therefore $\{x_n\}$ is a Cauchy sequence. There is $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(Tx_n, x^*) \\ &\leq \Big(\frac{d(x^*, Tx^*) + d(x_n, Tx_n)}{d(x^*, Tx^*) + d(x_n, Tx_n) + k}\Big) d(x_n, x^*) + d(Tx_n, x^*) \\ &\leq \Big(\frac{d(x^*, Tx^*) + d(x_n, x_{n+1})}{d(x^*, Tx^*) + d(x_n, x_{n+1}) + k}\Big) d(x_n, x^*) + d(x_{n+1}, x^*) \\ d(Tx^*, x^*) &\leq 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Therefore $d(x^*, Tx^*) = 0$. Thus, $Tx^* = x^*$.

Uniqueness

suppose x^* and y^* are two fixed points of T.

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \Big(\frac{d(x^*, Tx^*) + d(y^*, Ty^*)}{d(x^*, Tx^*) + d(y^*, Ty^*) + k}\Big) d(x^*, y^*) \\ &\leq 0 \\ d(x^*, y^*) &= 0 \\ \Rightarrow \quad x^* &= y^* \end{aligned}$$

Hence x^* is an unique fixed point of T.

(ii) $d(T^n x^*, x^*) = d(T^{n-1}(Tx^*), x^*) = d(T^{n-1}x^*, x^*) = d(T^{n-2}(Tx^*), x^*) \cdots = d(Tx^*, x^*) = 0$

Hence $T^n x^*$ converges to a fixed point, for all $x^* \in X$. □

Corollary 2.2 : Let (X, d) be a complete metric space. Suppose the mapping $T : X \to X$ satisfies the following conditions:

$$d(Tx, Ty) \leq \Big(\frac{d(x, Tx) + d(y, Ty)}{d(x, Tx) + d(y, Ty) + 1}\Big) d(x, y) \tag{2}$$

for all $x, y \in X$. Then

(i) T has unique fixed point in X.

(ii) $T^n x^*$ converges to a fixed point, for all $x^* \in X$.

Proof : The proof of the corollary immediate by taking $k = 1$ in the above theorem. □

Theorem 2.3 : Let (X, d) be a complete metric space and let T be a mapping from X into itself. Suppose that T satisfies the following condition:

$$d(Tx,Ty) \leq \Big(\frac{d(y,Ty)}{d(x,Tx)+d(y,Ty)+k}\Big)d(x,y) \tag{3}$$

for all $x, y \in X$, where $k \geq 1$. Then

(i) T has unique fixed point in X.

(ii) $T^n x^*$ converges to a fixed point, for all $x^* \in X$.

Proof : (i) Let $x_0 \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that x_{n+1}=Tx_n. We have

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \Big(\frac{d(x_{n-1},Tx_{n-1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+k}\Big)d(x_n,x_{n-1}) \\ &\leq \Big(\frac{d(x_{n-1},x_n)}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+k}\Big)d(x_n,x_{n-1}) \\ &\leq \Big(\frac{d(x_{n-1},x_n)}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+k}\Big)d(x_n,x_{n-1}). \end{aligned}$$

Take

$$\beta_n = \frac{d(x_{n-1},x_n)}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+k},$$

we have

$$\begin{aligned} d(x_{n+1},x_n) &\leq \beta_n d(x_n,x_{n-1}) \\ &\leq (\beta_n\beta_{n-1})d(x_{n-1},x_{n-2}) \\ &\vdots \\ d(x_{n+1},x_n) &\leq (\beta_n\beta_{n-1}\cdots\beta_1)d(x_1,x_0). \end{aligned}$$

Observe that (β_n) is non increasing, with positive terms.
So, $\beta_1 ... \beta_n \leq \beta_1^n$ and $\beta_1^n \to 0$. It follows that

$$\lim_{n\to\infty}(\beta_1\beta_2\cdots\beta_n) = 0.$$

Thus, it is verified that

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0.$$

Now for all $m, n \in \mathbb{N}$ and $m > n$ we have

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\leq [(\beta_n \beta_{n-1} \cdots \beta_1) + (\beta_{n+1} \beta_n \cdots \beta_1) + \cdots + (\beta_{m-1} \beta_{m-2} \cdots \beta_1)] d(x_1, x_0) \\
d(x_m, x_n) &= \sum_{k=n}^{m-1} (\beta_k \beta_{k-1} \cdots \beta_1) d(x_1, x_0)
\end{aligned}$$

where, $a_k = (\beta_k \beta_{k-1} \cdots \beta_1)$.
Now, $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, $\sum\limits_{k=1}^{\infty} a_k$ is finite and $\sum\limits_{k=n}^{m-1} (\beta_k \beta_{k-1} \cdots \beta_1) \to 0$, as $m, n \to \infty$.
Hence $\{a_k\}$ is convergent by D'Alembert's ratio test.
Therefore $\{x_n\}$ is a Cauchy sequence. There is $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

$$\begin{aligned}
d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(Tx_n, x^*) \\
&\leq \Big(\frac{d(x_n, Tx_n)}{d(x^*, Tx^*) + d(x_n, Tx_n) + k}\Big) d(x_n, x^*) + d(Tx_n, x^*) \\
&\leq \Big(\frac{d(x_n, x_{n+1})}{d(x^*, Tx^*) + d(x_n, x_{n+1}) + k}\Big) d(x_n, x^*) + d(x_{n+1}, x^*) \\
d(Tx^*, x^*) &\leq 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

Therefore $d(x^*, Tx^*) = 0$. Thus, $Tx^* = x^*$.

Uniqueness

suppose x^* and y^* are two fixed points of T.

$$\begin{aligned}
d(x^*, y^*) &= d(Tx^*, Ty^*) \\
&\leq \Big(\frac{d(y^*, Ty^*)}{d(x^*, Tx^*) + d(y^*, Ty^*) + l}\Big) d(x^*, y^*) \\
&\leq 0 \\
d(x^*, y^*) &= 0 \\
&\Rightarrow \quad x^* = y^*
\end{aligned}$$

Hence x^* is an unique fixed point of T.
(ii) $d(T^n x^*, x^*) = d(T^{n-1}(Tx^*), x^*) = d(T^{n-1} x^*, x^*) = d(T^{n-2}(Tx^*), x^*) \ \cdots = d(Tx^*, x^*) = 0$

Hence $T^n x^*$ converges to a fixed point, for all $x^* \in X$. □

Corollary 2.4 : Let (X, d) be a complete metric space and let T be a mapping from X into itself. Suppose that T satisfies the following condition:

$$d(Tx,Ty) \le \Big(\frac{d(y,Ty)}{d(x,Tx)+d(y,Ty)+1}\Big)d(x,y) \tag{4}$$

for all $x, y \in X$. Then

(i) T has unique fixed point in X.

(ii) $T^n x^*$ converges to a fixed point, for all $x^* \in X$.

Proof : The proof of the corollary immediate by taking $k = 1$ in the above theorem. □

Theorem 2.5 : Let (X, d) be a complete metric space. Suppose the mapping $T : X \to X$ satisfies the following conditions:

$$d(Tx,Ty) \le \Big(\frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+k}\Big)(d(x,Tx)+d(y,Ty)) \tag{5}$$

for all $x, y \in X$, where $k \ge 1$. Then

(i) T has unique fixed point in X.

(ii) $T^n x^*$ converges to a fixed point, for all $x^* \in X$.

Proof : (i) Let $x_0 \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that $x_{n+1}{=}Tx_n$.

$$\begin{aligned}
d(x_n,x_{n+1}) &= d(Tx_n,Tx_{n-1})\\
&\le \Big(\frac{d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)}{d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1})+k}\Big)(d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1}))\\
&\le \Big(\frac{d(x_n,x_n)+d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+k}\Big)(d(x_n,x_{n+1})+d(x_n,x_{n-1}))\\
&\le \Big(\frac{d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+k})(d(x_n,x_{n+1})+d(x_n,x_{n-1}))\\
d(x_n,x_{n+1}) &\le \Big(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+k}\Big)(d(x_n,x_{n+1})+d(x_n,x_{n-1}))
\end{aligned}$$

Take

$$\beta_n = \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+k},$$

we have

$$\begin{aligned}
d(x_{n+1},x_n) &\leq \beta_n(d(x_n,x_{n+1})+d(x_n,x_{n-1}))\\
(1-\beta_n)d(x_{n+1},x_n) &\leq \beta_n d(x_n,x_{n-1})\\
d(x_{n+1},x_n) &\leq \frac{\beta_n}{(1-\beta_n)}d(x_n,x_{n-1})\\
&\leq \frac{\beta_n\beta_{n-1}}{(1-\beta_n)(1-\beta_{n-1})}d(x_{n-1},x_{n-2})\\
&\vdots\\
&\leq \frac{\beta_n\beta_{n-1}\cdots\beta_1}{(1-\beta_n)(1-\beta_{n-1})\cdots(1-\beta_1)}d(x_1,x_0).\\
d(x_{n+1},x_n) &\leq \gamma_n d(x_1,x_0)
\end{aligned}$$

where

$$\gamma_n=\frac{\beta_n\beta_{n-1}\cdots\beta_1}{(1-\beta_n)(1-\beta_{n-1})\cdots(1-\beta_1)}$$

Observe that (β_n) is non increasing, with positive terms. So, $\beta_1...\beta_n \leq \beta_1^n$ and $\beta_1^n \to 0$. It follows that

$$\lim_{n\to\infty}(\beta_1\beta_2\cdots\beta_n)=0.$$

Therefore

$$\lim_{n\to\infty}\gamma_n=0$$

Thus, it is verified that

$$\lim_{n\to\infty}d(x_{n+1},x_n)=0$$

Now for all $m,n\in\mathbb{N}$ and $m>n$ we have

$$\begin{aligned}
d(x_m,x_n) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m)\\
&\leq [\gamma_n+\gamma_{n+1}+\cdots+\gamma_{m-1}]d(x_1,x_0)\\
d(x_m,x_n) &\leq \sum_{k=n}^{m-1}\gamma_k d(x_1,x_0)
\end{aligned}$$

Where $a_k=\gamma_k$.

Now, $\lim\limits_{k\to\infty}\frac{a_{k+1}}{a_k}<1,\ \sum\limits_{k=1}^{\infty}a_k$ is finite and $\sum\limits_{k=n}^{m-1}\gamma_k\to 0,$ as $m,n\to\infty$.

Hence $\{\gamma_k\}$ is convergent, by D'Alembert's ratio test.

Therefore $\{x_n\}$ is a Cauchy sequence. There is $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(Tx_n, x^*) \\ &\leq \Big(\frac{d(x^*, Tx_n) + d(x_n, Tx^*)}{d(x^*, Tx_n) + d(x_n, Tx^*) + k}\Big) d(x_n, x^*) + d(Tx_n, x^*) \\ &\leq \Big(\frac{d(x^*, x_{n+1}) + d(x_n, Tx^*)}{d(x^*, x_{n+1}) + d(x_n, Tx^*) + k}\Big) d(x_n, x^*) + d(x_{n+1}, x^*) \\ d(Tx^*, x^*) &\leq 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore $d(x^*, Tx^*) = 0$. Thus, $Tx^* = x^*$.

Uniqueness

suppose x^* and y^* are two fixed points of T.

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \Big(\frac{d(x^*, Ty^*) + d(y^*, Tx^*)}{d(x^*, Tx^*) + d(y^*, Ty^*) + k}\Big)(d(x^*, Tx^*) + d(y^*, Ty^*)) \\ &\leq 0 \\ d(x^*, y^*) &= 0 \\ \Rightarrow \quad x^* &= y^* \end{aligned}$$

Hence x^* is an unique fixed point of T.

(ii) $d(T^n x^*, x^*) = d(T^{n-1}(Tx^*), x^*) = d(T^{n-1}x^*, x^*) = d(T^{n-2}(Tx^*), x^*) \cdots = d(Tx^*, x^*) = 0$

Hence $T^n x^*$ converges to a fixed point, for all $x^* \in X$.

Corollary 2.6 : Let (X, d) be a complete metric space. Suppose the mapping $T : X \to X$ satisfies the following conditions:

$$d(Tx, Ty) \leq \Big(\frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1}\Big)(d(x, Tx) + d(y, Ty)) \tag{6}$$

for all $x, y \in X$. Then

(i) T has unique fixed point in X.

(ii) $T^n x^*$ converges to a fixed point, for all $x^* \in X$.

Proof : The proof of the corollary immediate by taking $k = 1$ in the above theorem. □

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