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MENGERS THEOREM AND LABELED GRAPHS

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Abstract

Our aim is to apply graph theory to SNS and in this pursuit we found that various theorems on labeled graphs are most suitable. In this connection we examine mengers theorem and extend this to labeled graphs, so as to use this theorem to study SNS. In the process of extending mengers theorem we define some new concepts like Semi labeled graphs, Mixed Labeled graphs, etc., and for this purpose we recall some standard definitions.

1. Introduction

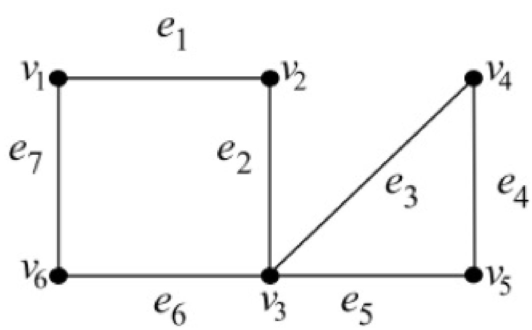
In our attempt to apply this theorem in face book we found that the theorem is not applicable as such in face book, since Labeled graphs find applications in face book. Hence we require extending mengers theorem to labeled graphs. We state mengers theorem in terms of labeled graphs and verify it in some cases and then apply it to face book.

Key Words : *Labeled graphs, Mengers theorem, Semi labeled graphs, Mixed labeled graphs.*

Definitions[reference:2]

Path, Trial: A sequence $v_0, e_1, v_1, e_2, \cdots, e_n v_n$ is called a path if vertices are distinct, if edges are distinct it is called a trial.

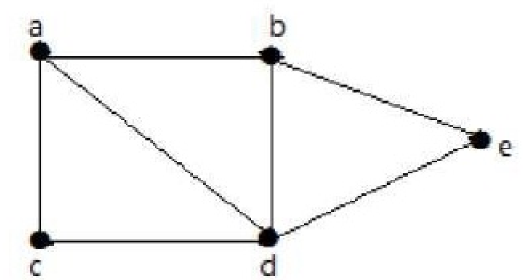
Example : Here v_1e_1 v_2e_2 v_3e_3 v_4e_4 is a **path** from vertex V_1 to vertex V_5 and v_1e_1 v_1e_1 v_2e_2 v_3e_6 $v_6e_7v_1$ is a trial.



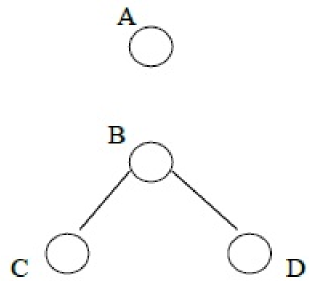
Connected/Disconnected Graph:[reference:2]

If there is a path joining any pair of vertices in a graph then the graph is called a Connected graph otherwise it is called Disconnected.

Example : In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'.



Example :

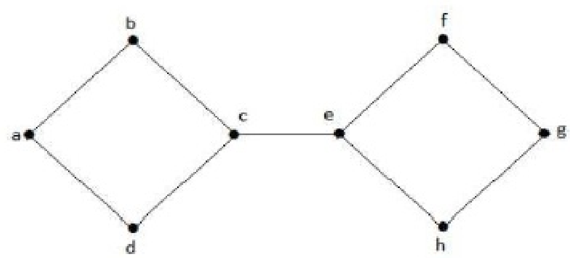


In the example above traversing from vertex A to a vertex B is not possible because there is no path between them directly or indirectly. Hence it is disconnected.

Cut Edge/Cut Vertex (Bridge): [reference:1 and 3]

An edge e(vertex v) is called a cut edge(cut vertex) if $E - \{e\}$ or $V - \{v\}$ results in a disconnected graph.

Example : In the following graph, the cut edge is [(c, e), removal of the edge (c,e) disconnects the given graph.

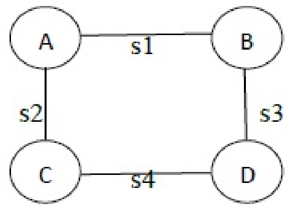


Label , Vertex /Edge labeled Graphs:[reference:5]

Label :

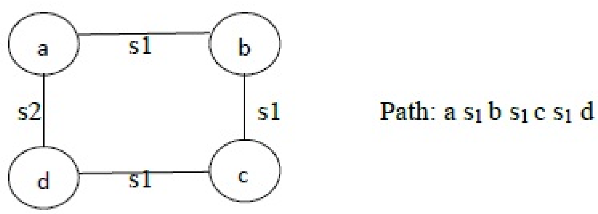
A label in a graph is a symbol assigned to its edge/vertex. Such graphs are called Labeled Vertex/Edge graphs. A graph is called a labeled graph if it is vertex/edge labeled.

Example :



Labeled Path : A labeled path in an edge labeled graph is an usual path whose edges have the same label.

Example :

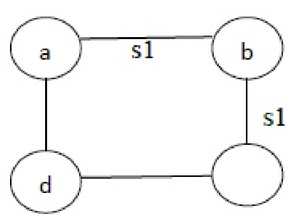


Semi Labeled Edge/vertex graph :

A graph in which some edges/vertices are labeled is called a Semi Labeled edge/vertex graph.

A graph is semi labeled if it is semi labeled edge/vertex graph.

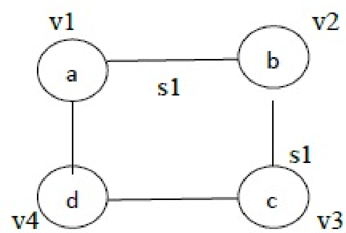
Example :



Semi Labeled path :

A Semi Labeled path in a semi labeled edge graph is an usual path where only a few edges of whole have labels.

Example :

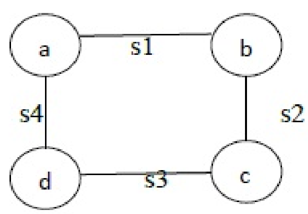


Semi labeled Path a, s_1, b, s_1, c.

Mixed Labeled Edge/vertex graph :

A labeled graph in which all the edges/vertices have different labels is called a Mixed Labeled edge/vertex graph.

Example :

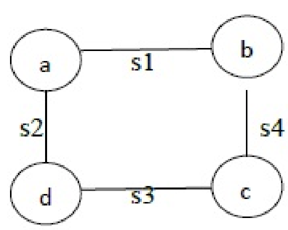


Mixed labeled Path : a, s_1, b, s_2, c.

Mixed Labeled path :

A Mixed Labeled path in a labeled edge graph is an usual path whose edges have different labels.

Example:



Path : a s_1 b s_4 c s_3 d

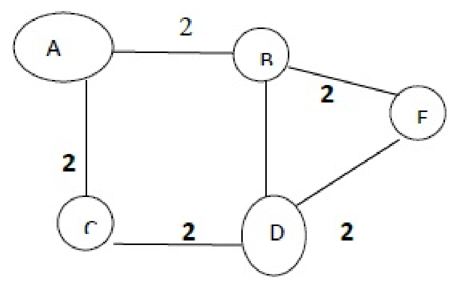
NOTE : In a Semi labeled edge graph with an edge without label is not a part of a labeled path.

NOTE : A Labeled Path with label say M does not contain an edge with a label other than M.

Labeled Connected/Disconnected Graph :

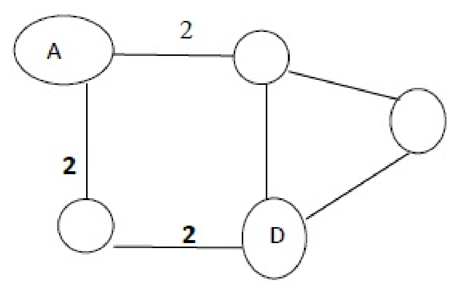
If there is a labeled path joining any pair of vertices in a graph then the graph is called a Labeled Connected graph otherwise it is called Labeled Disconnected.

Example :



The above graph is a Labeled connected graph.

Example :

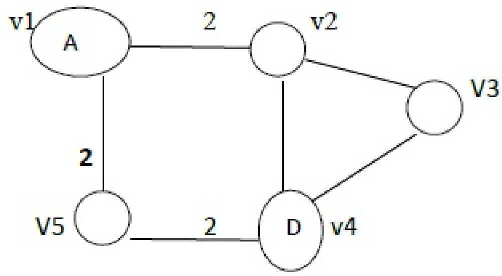


The above graph is a Labeled disconnected graph.

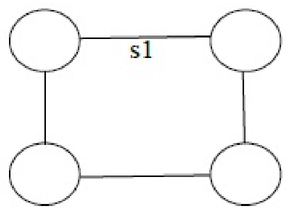
Semi Connected/Disconnected Graph :

If there is a Semi labeled path joining any pair of vertices in a graph then the graph is called a Semi Connected graph otherwise it is called Semi Disconnected.

Example :



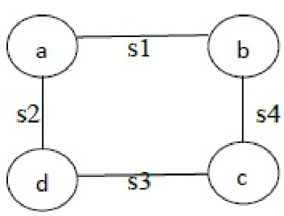
Example :



Mixed Connected/Disconnected Graph :

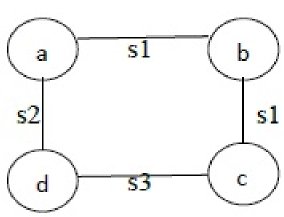
If there is a Mixed labeled path joining any pair of vertices in a graph then the graph is called a Mixed Connected graph otherwise it is called Mixed Disconnected.

Example :



Path : a s_1 b s_4 c s_3 d

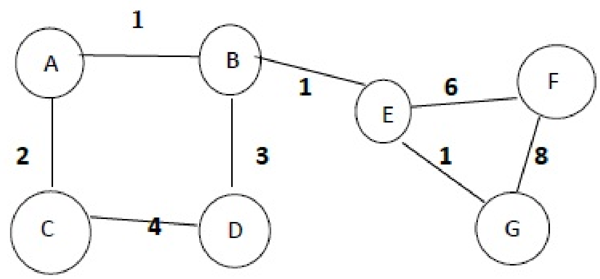
Example :



Path : a s_1 b s_4 c s_3 d

Labeled cut edge : In an edge labeled graph a labeled edge cut is a label of an edge whose removal results in a disconnected graph.

Example :

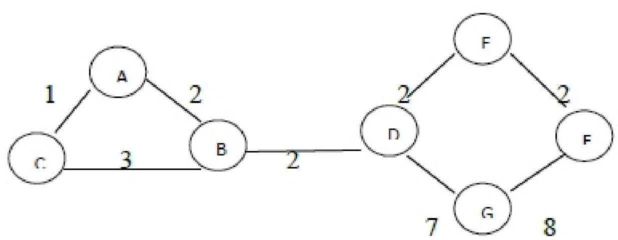


Path : A 1 B 1 E 1 G, therefore removal of the label 1 of the edge BE disconnects the above graph. Therefore removal of a label from an edge in an edge labeled graph results in a disconnected graph.

Note : Here a labeled cut edge is not a cut edge.

Difference between Labeled Disconnected Graph and Disconnected Graph :

Consider a graph G:

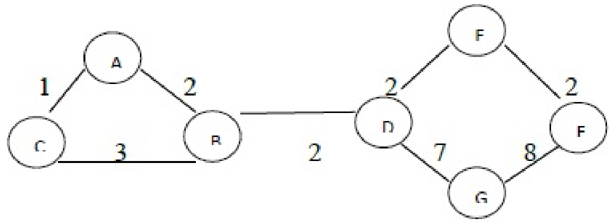


A Labeled path from A to F is : A 2 B 2 D 2 E 2 F.

Removal of the edge BD results in a disconnected graph, whereas removal of the label '2' from the edge BD results in a labeled disconnected graph.

Difference between SEMI Disconnected Graph and Disconnected Graph :

Consider a graph G:

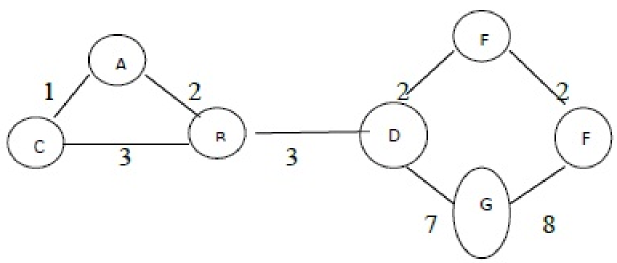


A Labeled path from A to F is : : A 2 B 2 D 7 G 8 F.

Removal of the edge BD results in a disconnected graph, whereas removal of the label '2' from the edge BD results in a Semi labeled disconnected graph.

Difference between MIXED Disconnected Graph and Disconnected Graph :

Consider a graph G:



A Labeled path from A to F is : A 2 B 3 D 7 G 8 F.

Removal of the edge BD results in a disconnected graph, whereas removal of the label '3' from the edge BD results in a Mixed disconnected graph.

Mengers Theorem[reference:4]

This theorem is a characterization of the connectivity of in finite undirected graphs in terms of the minimum number of disjoint paths that can be found between any pair of vertices.

Connectivity Number for Vertices : $K(G)$ is defined as a minimum number of vertices whose removal from G results in a disconnected graph or in the trivial graph.

Connectivity Number for Edges : $\lambda(G)$ is defined as a minimum number of edges whose removal from G results in a disconnected graph or in the trivial graph.

The Edge connectivity version of Mengers Theorem :

Let G be a finite undirected graph and X and Y two distinct vertices. Then the theorem states that the size of the minimum edge cut for X and Y (the minimum number of edges whose removal disconnects X and Y) is equal to the maximum number of pair wise edge independent paths from X to Y.

Labeled Connectivity Number for Edges : $L\ \lambda(G)$ is defined as a minimum number of labeled edges whose removal from G results in a disconnected graph.

SEMI Labeled Connectivity Number for Edges : $S\lambda(G)$ is defined as a minimum number of semi labeled edges whose removal from G results in a SEMI disconnected graph.

MIXED Labeled Connectivity Number for Edges : $M\lambda(G)$ is defined as a minimum number of mixed labeled edges whose removal from G results in a MIXED disconnected graph.

The Vertex connectivity version of Mengers Theorem :

Let G be a finite undirected graph and X and Y two distinct vertices. Then the theorem states that the size of the minimum vertex cut for X and Y (the minimum number of edges whose removal disconnects X and Y) is equal to the maximum number of pair wise vertex independent paths from X to Y.

Labeled Connectivity Number for Vertices : $LK(G)$ is defined as a minimum number of labeled vertices whose removal from G results in a disconnected graph.

SEMI Labeled Connectivity Number for Vertices : $SK(G)$ is defined as a minimum number of SEMI labeled vertices whose removal from G results in a SEMI disconnected graph.

MIXED Labeled Connectivity Number for Vertices : $MK(G)$ is defined as a minimum number of MIXED labeled vertices whose removal from G results in a MIXED disconnected graph.

We next show that Mengers theorem in terms of labeled edge graph is not true in the following **example**.

Menger's Theorem in terms of labeled edge graph :

Let G be a finite edge labeled graph and X and Y two distinct vertices. Then the theorem states that the size of the minimum labeled edge cut for X and Y is equal to the maximum number of pair wise labeled edge independent paths from X to Y.

Example : In this example a labeled graph is given by $G = \{E, V, L\}$.

The example uses induction on the number of labeled edges(m) under the following 3 cases:

Case (i) : Neither u nor v is labeled adjacent to each other.

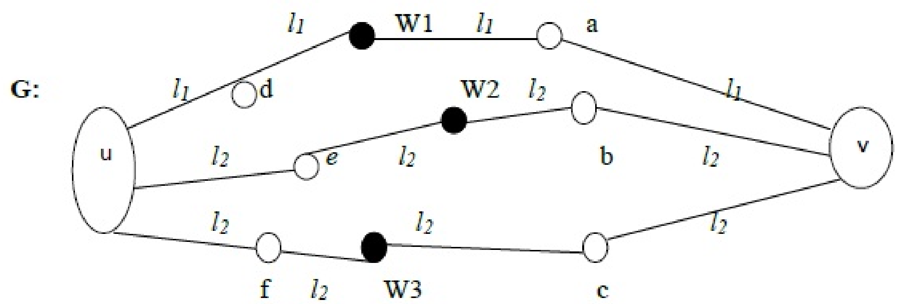
Case (ii) : Let W be a set of vertices which disconnects u and v. Either u or v is labeled adjacent to each vertex of W.

Case (iii) : Both u and v are labeled adjacent to each vertex of W.

Case 1 : Let $m = 3$.

Case (i) : Neither u nor v is labeled adjacent to each other.

Consider a labeled graph where the labeled edges are = 3.



In the above graph: The minimum labeled edge cut for u and $v = 3$.

The maximum number of pair wise labeled edge independent paths from u to $v = 2$.

Therefore the size of the minimum labeled edge cut for u and v is **NOT EQUAL** to the maximum number of pair wise labeled edge independent paths from u to v. Therefore case (i) does not hold true.

The Menger's theorem for labeled vertices is not true in general as seen in the above example, but under some conditions this theorem is true. i.e., when the number of labels given to the edges IS EQUAL TO the number of paths between u and v.

Menger's Theorem verified IN TERMS OF labeled edge graph : (our definition)

Let G be a finite edge labeled graph and X and Y two distinct vertices. Then the theorem states that the size of the minimum labeled edge cut for X and Y is equal to the maximum number of pair wise labeled edge independent paths from X to Y **only when the number of labels given to the edges IS EQUAL TO the number of paths between u and v.**

Example : In this example a labeled graph is given by $G = \{E, V, L\}$.

This example uses induction on the number of labeled edges(m) under the following 3 cases:

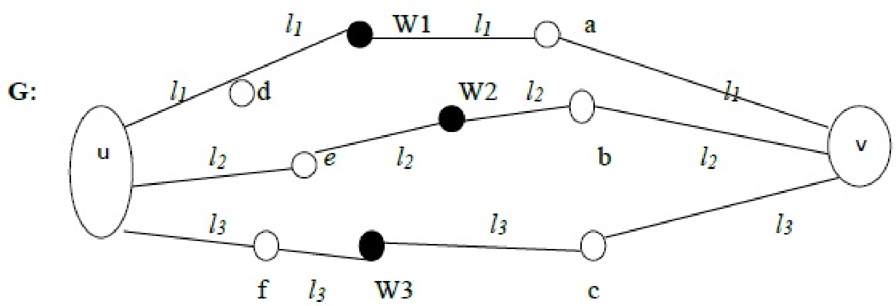
Case (i) : Neither u nor v is labeled adjacent to each other.

Case (ii) : Let W be a set of vertices which disconnects u and v. Either u or v is labeled adjacent to each vertex of W.

Case (iii) : Both u and v are labeled adjacent to each vertex of W.

Case 1 : Let $m = 3$.

Case (i) : Neither u nor v is labeled adjacent to each other. Consider a labeled graph where the labeled edges are $= 3$.

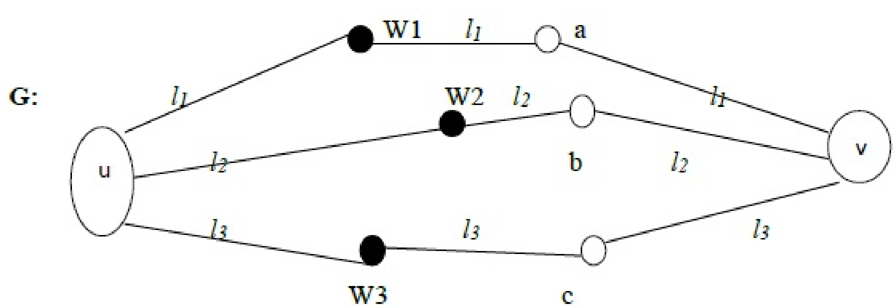


In the above graph: The minimum labeled edge cut for u and $v = 3$.

The maximum number of pair wise labeled edge independent paths from u to $v = 3$.

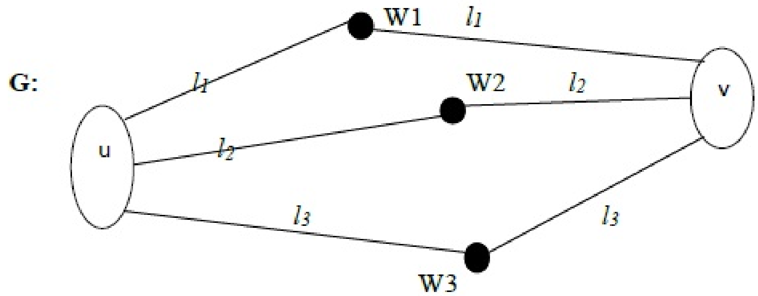
Therefore the size of the minimum labeled edge cut for u and v is EQUAL TO the maximum number of pair wise labeled edge independent paths from u to v.

Case (ii) : Let W be a set of vertices which disconnects u and v. Either u or v is labeled adjacent to each vertex of W. Consider a labeled graph where the labeled edges are $= 3$.



In the above graph: The minimum labeled edge cut for u and $v = 3$.
The maximum number of pair wise labeled edge independent paths from u to $v = 3$.
Therefore the size of the minimum labeled edge cut for u and v is **EQUAL TO** the maximum number of pair wise labeled edge independent paths from u to v.

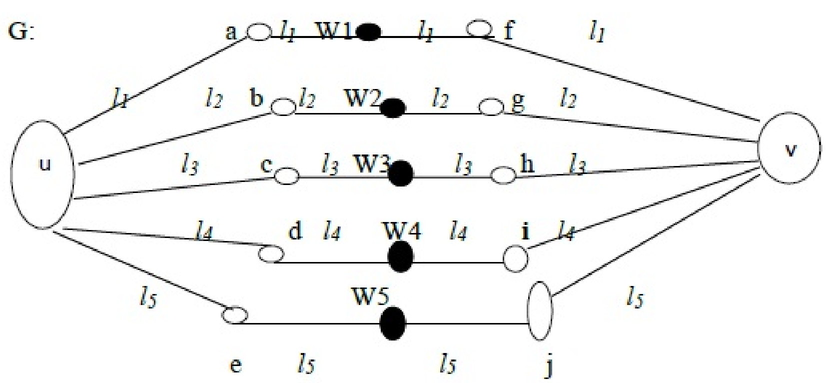
Case (iii) : Both u and v are labeled adjacent to each vertex of W.
Consider a labeled graph where the labeled edges are $= 3$.



In the above graph: The minimum labeled edge cut for u and $v = 3$.
The maximum number of pair wise labeled edge independent paths from u to $v = 3$.
Therefore the size of the minimum labeled edge cut for u and v is **EQUAL TO** the maximum number of pair wise labeled edge independent paths from u to v.

Case 2 : Let $m > 3$:

Case (i) : Neither u nor v is labeled adjacent to each other.
Consider a labeled graph where the labeled edges are $= 5$.



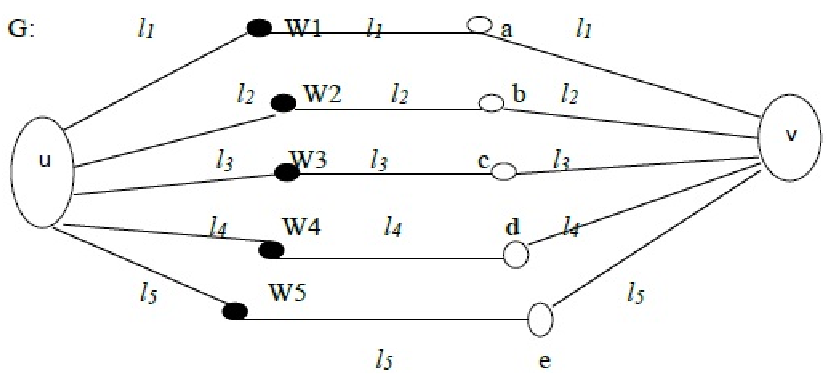
In the above graph : The minimum labeled edge cut for u and $v = 5$.

The maximum number of pair wise labeled edge independent paths from u to $v = 5$.

Therefore the size of the minimum labeled edge cut for u and v is **EQUAL TO** the maximum number of pair wise labeled edge independent paths from u to v.

Case (ii) : Let W be a set of vertices which disconnects u and v. Either u or v is labeled adjacent to each vertex of W.

Consider a labeled graph where the labeled edges are = 5



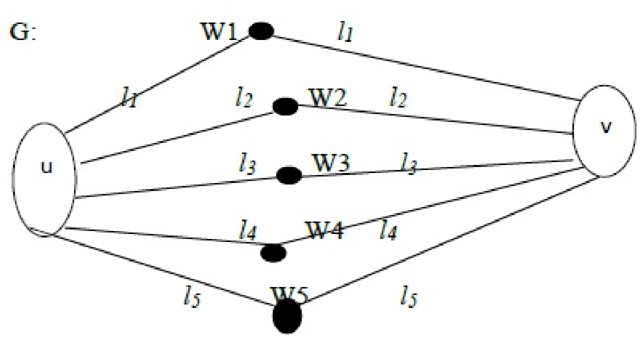
In the above graph: The minimum labeled edge cut for u and $v = 5$.

The maximum number of pair wise labeled edge independent paths from u to $v = 5$.

Therefore the size of the minimum labeled edge cut for u and v is **EQUAL TO** the maximum number of pair wise labeled edge independent paths from u to v.

Case (iii) : Both u and v are labeled adjacent to each vertex of W.

Consider a labeled graph where the labeled edges are = 5



In the above graph: The minimum labeled edge cut for u and $v = 5$.

The maximum number of pair wise labeled edge independent paths from u to $v = 5$. Therefore the size of the minimum labeled edge cut for u and v is **EQUAL TO** the maximum number of pair wise labeled edge independent paths from u to v.

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