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A COMPLEX AL-TEMEME TRANSFORM

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Abstract

In this reseach, it was reviewed a definition of complex Al-Tememe transformation and concluded that the transform of some famous function and how to get benefit from theorem presented in this reseach to find the inverse, and using this transform in solving some types of ordinary differential equations with variable coefficients as Euler equation and equation including the multiplication of polynomials and logarithms.

1. Introduction

Al-Tememe is anew transformation emerged in 2008 and used to solve some types of differentil equations. As well as it can be applied in some fields like physics, engineering and bio-medical signal processing New uses appear in 2015 for this transformation such as solving some kinds of ODE'S with variable coefficients and identify some of the basic things differentiation and integration, the unit step function, solving linear systems of ordinary differential equations, and convolution of Al-Tememe transformation.

Addionally, the usefulusage of Al-Tememe transformation in solving ODEand some application in physics and engineering.

Recently in 2016 solving some types of PDE'S with variable coefficients, how Al-Tememe transformation useful in some applications has been shows the differences between Al-Tememe transformation and fourier transformation in speech recognition and in analyzing gaussian noise and which one the better description of signals. Also, illustrating that AL-Tememe transformation approach is more efficiet in solving the laplace's equation in polar coordinates and comparing it with other mothods.

2. Preliminaries

Definition 1 [1] : Let f is defined function at interval (a, b) then the integral transform for f whose it's symbol $F(p)$ is defined as :

$$F(p) = \int_a^b k(p, x) f(x) dx,$$

where k is a function of two variables, called the kernel of the transform and a, b are real numbers or $\mp\infty$, such that the above integral converges.

Definition 2 : A complex Al-Tememe transform for the function $f(x); x > 1$ is defined by the following integral :

$$\mathcal{T}^c[f(x)] = \int_1^\infty x^{-ip} f(x) dx = F(ip),$$

such that this integral is convergent in $(1, \infty)$, p is positive constant and x^{-ip} is the kernel of complex Al-Tememe transform.

Definition 3 : Let $\mathcal{T}^c[(x)] = F(ip)$ represents a complex Al-Tememe transform of $f(x)$. Then (x) is said to be inverse of a complex Al-Tememe transform and it write

$$f(x) = \mathcal{T}^{c-1}(F(ip))/$$

$*$ A complex Al-Tememe transform of some famous functions

1.

$$\begin{aligned} T^c(1) &= \int_1^\infty x^{-ip} f(x) dx = \int_1^\infty x^{-ip} dx = \left. \frac{x^{-ip+1}}{-ip+1} \right|_1^\infty \\ &= 0 - \frac{1}{-ip+1} = \frac{1}{-1+ip} \times \frac{-1-ip}{-1-ip} = \frac{-1-ip}{1+p^2} \\ &= \frac{-1}{1+p^2} - \frac{p}{1+p^2} i \end{aligned}$$

2.

$$\begin{aligned}
T^c(x^n) &= \int_1^\infty x^{-ip}x^n dx = \int_1^\infty x^{n-ip}dx = \frac{x^{n-ip+1}}{n-ip+1}\Big|_1^\infty \\
&= 0 - \frac{1}{n-ip+1} = \frac{1}{ip-(n+1)} \times \frac{-ip-(n+1)}{-ip-(n+1)} = \frac{-ip-(n+1)}{p^2+(n+1)^2} \\
&= \frac{-(n+1)}{p^2+(n+1)^2} - \frac{p}{p^2(n+1)^2}i.
\end{aligned}$$

3.

$$\begin{aligned}
T^c(ln\ x) &= \int_1^\infty x^{ip} ln\ xdx \\
& u = ln\ x, \quad dv = x^{-ip}dx \\
& du = \frac{1}{x}dx, \quad v = \frac{x^{-ip+1}}{-ip+1} \\
&= ln\ x \cdot \frac{x^{-ip+1}}{-ip+1}\Big|_1^\infty - \int_1^\infty \frac{x^{-ip+1}}{-ip+1} \cdot \frac{1}{x}dx \\
&= 0 - 0 - \frac{1}{-ip+1}\left[\int_1^\infty x^{ip}dx\right] \\
&= \frac{1}{ip-1}\left[\frac{x^{-ip+1}}{-ip+1}\right]\Big|_1^\infty = \frac{1}{ip-1}\left[0 - \frac{1}{ip-1}\right] = \frac{1}{ip-1}\left[\frac{1}{ip-1}\right] \\
&= \frac{1}{(ip-1)^2} \times \frac{(-ip-1)^2}{(-ip-1)^2} = \frac{(-ip-1)^2}{(p^2+1)^2} \\
&= \frac{1-p^2}{(p^2+1)^2} + \frac{2p}{(p^2+1)^2}i.
\end{aligned}$$

4. $T^c(\sinh alnx)$, a constant

$$\begin{aligned}
T^c(\sinh alnx) &= T\left(\frac{e^{annx} - e^{-alnx}}{2}\right) \\
&= \frac{1}{2}T^c(x^a - x^{-a}) = \frac{1}{2}\left[\frac{1}{ip-(a+1)} - \frac{1}{ip-(-a+1)}\right]. \\
&= \frac{1}{2}\left[\frac{ip+1-1-ip+a+1}{(ip-a-1)(ip+a-1)}\right] = \frac{2a}{2((ip-1)-a)((ip-1)+a)} \\
&= \frac{a}{(ip-1)^2-a^2} \times \frac{(-ip-1)^2-(a)^2}{(-ip-1)^2-(a)^2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{a[(-ip-1)^2-a^2]}{(p_2+1)^2+a^4} = \frac{a((-p^2+1-a^2)+2ip)}{(p^2+1)^2+a^4} \\
&= \frac{a-(ap^2+a^3)}{(p^2+1)^2+a^4} + \frac{2ap}{(p^2+1)^2+a^4} i \\
&= \frac{-a[(p^2+1)+a^2]}{(p^2+1)^2+a^4} + \frac{2ap}{(p^2-1)^2+a^4} i.
\end{aligned}$$

5. $T^c(\cosh\ alnx)$, a constant.

$$\begin{aligned}
T^c(\cosh alnx) &= T^c\left(\frac{e^{alnx}-e^{-alnx}}{2}\right) \\
&= \frac{1}{2}\left[\frac{1}{ip-(a+1)} + \frac{1}{ip-a+1}\right] = \frac{1}{2}\left[\frac{ip+a-1+ip-a-1}{(ip-1-a)(ip-1+a)}\right] \\
&= \frac{1}{2}\left[\frac{2ip-2}{(ip-1)^2-a^2}\right] = \frac{ip-1}{(ip-1)^2-a^2} \times \frac{(-ip-1)^2-a^2}{(-ip-1)^2-a^2} \\
&= \frac{(ip-1)((-ip-1)^2-a^2)}{(p^2+1)^2+a^4} \\
&= \frac{(-p^2+1-a^2+2ip)(ip-1)}{(p^2+1)^2+a^4} \\
&= \frac{-ip^3+ip-ipa^2-2p^2+p^2-1+a^2-2ip}{(p^2+1)^2+a^4} \\
&= \frac{-p^2+a^2-1}{(p^2+1)^2+a^4} - \frac{(p^3+a^2p+p)}{(p^2+1)^2+a^4} i \\
&= \frac{-(p^2+1)+a^2}{(p^2+1)^2+a^4} - \frac{p((p^2+1)+a^2)}{(p^2+1)^2+a^4} i.
\end{aligned}$$

6. $T^c(\sin\ alnx)$, a constant.

$$\begin{aligned}
T^c(\sin\ alnx) &= T\left(\frac{e^{ialnx}-e^{-ialnx}}{2}\right) = \frac{1}{2i} T^c(x^{ia}-x^{-ia}) \\
&= \frac{1}{2i}\left[\frac{1}{ip-(ia+1)} - \frac{1}{ip-(-ia+1)}\right] = \frac{1}{2l}\left[\frac{ip+ia-1-ip+ia+1}{(ip-ia-1)(ip+ia-1)}\right] \\
&= \frac{2ia}{2i(ip-1)-ia)((ip-1)+ia)} = \frac{a}{(ip-1)^2-a^2} \times \frac{(-ip-1)^2+a^2}{(-ip-1)^2+a^2} \\
&= \frac{a[(-ip-1)^2+a^2]}{(p^2+1)^2+a^4} = \frac{a((-p^2+1+a^2)+2ip)}{(p^2+1)^2+a^4} \\
&= \frac{a+a^3-ap^2}{(p^2+1)^2+a^2} + \frac{2ap}{(p^2+1)^2+a^4} i \\
&= \frac{-a[(p^2-1)-a^2]}{(p^2+1)^2+a^4} + \frac{2ap}{(p^2+1)^2+a^4} i.
\end{aligned}$$

7. $T^c(\cosh\ alnx)$, a constant.

$$\begin{aligned}
T^c(\cosh alnx) &= T^c\left(\frac{e^{ialnx}-e^{-ialnx}}{2}\right)=\frac{1}{2}T^c(x^{ia}+x^{-ia})\\
&= \frac{1}{2}\left[\frac{1}{ip-(ia+1)}+\frac{1}{ip-(-ia+1)}\right]=\frac{1}{2}\left[\frac{ip+a-1+ip-ia-1}{(ip-1-ia)(ip-1+ia)}\right]\\
&= \frac{1}{2}\left[\frac{2ip-2}{(ip-1)^2+a^2}\right]=\frac{ip-1}{(ip-1)^2+a^2}\times\frac{(-ip-1)^2+a^2}{(-ip-1)^2+a^2}\\
&= \frac{(ip-1)((-ip-1)^2+a^2)}{(p^2+1)^2+a^4}\\
&= \frac{(-p^2+1+a^2+2ip)(ip-1)}{(p^2+1)^2+a^4}\\
&= \frac{-ip^3+ip+ipa^2-2p^2+p^2-1+a^2-2ip}{(p^2+1)^2+a^4}\\
&= \frac{-p^2+a^2+1}{(p^2+1)^2+a^4}+\frac{(-p^3+a^2p-p)}{(p^2+1)^2+a^4}i\\
&= \frac{[-(p^2-1)-a^2]}{(p^2+1)^2+a^4}-\frac{p((p^2+1)-a^2)}{(p^2+1)^2+a^4}i.
\end{aligned}$$

Example 1 : Find acomplex Al-Tememe transform.

1.

$$-T^c(x^3)=\frac{-4}{p^2+16}-\frac{p}{p^2+16}i$$

2.

$$-T^c(-4lnx)=\frac{-4+4p^2}{(p^2+1)^2}-\frac{8p}{(p^2+1)^2}i$$

3.

$$\begin{aligned}
-T^c(\cosh(3lnx)) &= \tfrac{-p^2+9-1}{(p^2+1)^2+81}-\tfrac{p^3+9p+p}{(p^2+1)^2+81}i\\
&= \tfrac{-o^2+8}{(p^2+1)^2+81}-\tfrac{p^2+10p}{(p^2+1)^2+81}i
\end{aligned}$$

4.

$$\begin{aligned}
-T^c(\sin(-4lnx)) &= \tfrac{-4+64-4p^2}{(p^2+1)^2+256}+\tfrac{8p}{(p^2+1)^2+256}i\\
&= \tfrac{60-4p^2}{(p^2+1)^2+256}+\tfrac{8p}{(p^2+1)^2+256}i.
\end{aligned}$$

Theorem 1 : If $Tf(x)=F(ip)$ then

$$T\left(\int_1^x f(t)dt\right)=\frac{-1-ip}{1+p^2}T\{xf(x)\}.$$

Proof : Llet

$$\int_1^x f(t)dt = g(x), \quad g'(x) = f(x), \quad g(1) = 0$$

$$T\left(\int_1^x f(t)dt\right) = T(g(x)) = \int_1^\infty x^{-ip} g(x)dx$$

$$u = g(x) \qquad dv = x^{-ip}dx$$

$$\begin{aligned} du &= g'(x)dx \quad v = \frac{x^{-ip+1}}{-up+1} = \frac{1+ip}{1+p^2}x^{-ip+1} \\ &= \frac{1+ip}{1+p^2}x^{-ip+1}g(x)\Big|_1^\infty - \int_1^\infty \frac{1+ip}{1+p^2}x^{-ip+1}g'(x)dx \\ &= \frac{-1-ip}{1+p^2}\int_1^\infty x^{-ip}xg'(x)dx. \end{aligned}$$

$$T(g(x)) = \frac{-1-ip}{1+p^2}T[xf(x)] \text{ so } g(x) = T^{-1}\left[\frac{-1-ip}{1+p^2}T[xf(x)]\right].$$

Example 2 :

$$\begin{aligned} &T^{-1}\left[\frac{-p^2+5ip+4}{p^4+17p^2+16}\right] = T^{-1}\left[\frac{(-1-ip)(-4-ip)}{(p^2+1)(p^2+16)}\right] \\ &= T^{-1}\left[\frac{(-1-ip)}{(p^2+1)}T(x^3)\right] = T^{-1}\left[\frac{(-1-ip)}{(p^2+1)}T(x\ x^2)\right] \\ &= \int_1^x f(t)dt = \int_1^x t^2 dt = \frac{t^3}{3} = \frac{x^3}{3} - \frac{1}{3}. \end{aligned}$$

Example 3 :

$$\begin{aligned} T^{-1} &= \left[\frac{-ip^3+4p^2-ip+6}{(p^4+5p^2+4)+(p^2+9)}\right] = T^{-1}\left[\frac{(-p^2+ip-2)(ip-3)}{(p^2+1)(p^2+4)(p^2+9)}\right] \\ &= T^{-1}\left[\frac{(-1-ip)(2-ip)(-3-ip)}{(p^2+1)(p^2+4)(p^2+9)}\right] = T^{-1}\left[\frac{(-1-ip)}{(p^2+1)}\left(\frac{-p^2-ip-6}{(p^2+4)(p^2+9)}\right)\right] \end{aligned}$$

$$\begin{aligned} \frac{-p^2-ip-6}{(p^2+4)(p^2+9)} &= \frac{Aip+B}{(p^2+4)} + \frac{Cip+D}{(p^2+9)} \\ &= \frac{Aip^3+9Aip+Bp^2+9B+Cip^3+4Cip+Dp^2+4D}{(p^2+4)(p^2+9)} \end{aligned}$$

$$A + C = 0 \tag{1}$$

$$B + D = -1 \tag{2}$$

$$9A + 4C = 1 \tag{3}$$

$$9B + 4D = 6. \tag{4}$$

From (1) and (3), $A = \frac{-1}{5}, \quad C = \frac{1}{5}$.

From (2) and (4), $B = \frac{-2}{5}, \quad D = \frac{-3}{5}$.

$$\begin{aligned}
\frac{-p^2 - ip - 6}{(p^2+4)(p^2+9)} &= \frac{\frac{-1}{5}ip - \frac{2}{5}}{(p^2+4)} + \frac{\frac{1}{5}ip - \frac{3}{5}}{(p^2+9)} \\
&= \frac{1}{5}\left(\frac{-ip-2}{(p^2+4)}\right) + \frac{1}{5}\left(\frac{ip-3}{(p^2+9)}\right) = T\left(\frac{1}{5}x - \frac{1}{5}x^{-4}\right) \\
&= T^{-1}\left[\frac{-1-ip}{(p^2+1)}T\left(x\left(\frac{1}{5} - \frac{1}{5}x^{-5}\right)\right)\right] \\
&= \int_1^x f(t)dt = \int_1^x \frac{1}{5} - \frac{1}{5}t^{-5}dt \\
&= \frac{1}{5}\left[t - \frac{t^{-4}}{-4}\right] \\
&= \frac{1}{5}\left[\left(x - \frac{x^{-4}}{-4}\right) - \left(1 + \frac{1}{4}\right)\right] = \frac{x}{5} + \frac{x^{-4}}{20} - \frac{1}{4}.
\end{aligned}$$

Example 4 :

$$\begin{aligned}
&T^{-1}\left[\frac{(-1-ip)(-p^2+6pip+18)}{(p^2+1)(p^4+18p^2+162)}\right] \\
&= T^{-1}\left[\frac{(-1-ip)}{(p^2+1)}\left(\frac{(-p^2+6ip+18)}{(p^4+18p^2+162)}\right)\right] \\
&= T^{-1}\left[\frac{(-1-ip)}{(p^2+1)}\left(\frac{(-ip-3)^2+9}{(p^2+9)^2+81}\right)\right] \\
&= T^{-1}\left[\frac{(-1-ip)}{(p^2+1)}T(x^2 \sin 3lnx)\right] \\
&= T^{-1}\left[\frac{(-1-ip)}{(p^2+1)}T(x\ x\ \ \sin 3lnx)\right] = \int_1^x f(t)dt = \int_1^x t\ \ \sin 3lnt\ dt \\
&u = \sin 3lnt, \qquad du = \cos 3lnt\frac{3}{t}dt
\end{aligned}$$

$$dv = t\ dt, \qquad v = \frac{t^2}{2}$$
$$= \frac{t^2}{2}\sin 3lnt\Big|_1^x - \int_1^x \frac{t^2}{2}\cos 3lnt\ \frac{3}{t}dt$$
$$= \left(\frac{x^2}{2}\sin 3lnx - 0\right) - \frac{3}{2}\int_1^x t\ \cos 3lnt\ dt$$
$$u = \cos\ 3lnt \qquad du = -\sin 3lnt\frac{3}{t}dt$$
$$dv = tdt \qquad v = \frac{t^2}{2}$$
$$= \frac{x^2}{2}\sin 3lnx - \frac{3}{2}\left[\frac{t^2}{2}\cos 3lnt\Big|_1^x - \int_1^x \frac{t^2}{2}(-\sin 3lnt)\frac{3}{t}dt\right]$$
$$= \frac{x^2}{2}\sin 3lnx - \frac{3}{4}(x^2\cos 3lnx) + \frac{3}{4}x^2 - \frac{9}{4}\int_1^x t\sin 3lnt\ dt$$
$$\int_1^x t\sin 3lnt\ dt + \frac{9}{4}\int_1^x t\sin 3lnt\ dt$$
$$= \frac{x^2}{2}\sin 3lnx - \frac{3}{4}(x^2\cos 3lnx) + \frac{3}{4}x^2$$
$$= \frac{4}{13}\left[\frac{x^2}{2}\sin 3lnx - \frac{3}{4}(x^2\cos 3lnx) + \frac{3}{4}x^2\right].$$

Use acomplex Al-Tememe transform in solving some differential equations

$$\begin{aligned}
T(xy') &= \int_1^\infty x^{-ip}xy'\ \ dx = \int_1^\infty x^{ip+1}y'dx \\
u &= x^{-ip+1} \qquad\qquad dv = y'dx \\
du &= (-ip+1)x^{-ip}dx \qquad\qquad v = y \\
&= x^{-ip+1}y - \int_1^\infty (-ip+1)x^{-ip}ydx \\
&= 0 - y(1) - (-ip+1)\int_1^\infty x^{-ip}ydx \\
&= -y(1) + (ip-1)T(y) \\
T(x^2y'') &= \int_1^\infty x^{-ip}x^2y''\ \ dx = \int_1^\infty x^{-ip+2}y''dx \\
u &= x^{-ip+2} \qquad\qquad dv = y''dx
\end{aligned}$$

$$\begin{aligned} du &= (-ip+2)x^{-ip+1}dx \qquad\qquad v = y' \\ &= x^{-ip+2}y' - \int_1^\infty (-ip+2)x^{-ip+1}y'dx \\ &= 0 - y'(1) - (-ip+2)\int_1^\infty x^{-ip}x\ y'dx \\ &= -y'(1) + (ip-2)T(x\ y') \\ &= -y'(1) - (ip-2)y(1) + (ip-2)(ip-1)T(y) \end{aligned}$$

$$\begin{aligned} T(x^3y''') &= -y'' - (ip-3)y'(1) - (ip-3)(ip-2)y(1) + *ip-3)(ip-2)(ip-1)T(y) \\ T_n^x y^{(n)}) &= -y^{n-1} - (ip-n)y^{n-2} - (ip-n)(ip-(n-1)y^{n-3} - (ip-n)(ip-(n-1) \\ &\quad \cdots (ip-2)y(1) + -(ip-n)(ip-(n-1)\cdots(ip-2)(ip-1)T(y). \end{aligned}$$

The New Stylefor solving ordinary Differental Equation by using complex Altememe transform

Suppose we have a linear ordinary differential equation of order (n) with variables coefficients and due to certain initial conditions, which general form can be written as follow:

$$a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy' + a_ny = (x) \tag{1}$$

where $a_0, a_1, \cdots, a_n$ are constants, $y^{(n)}$ is the nth derivative of the function $y(x), f(x)$ is a continuous function whose complex Al-Tememe transform can be determined and $y(1), \cdots, y^{(n-1)}(1)$ are defined.

To find a solution of DE(1) we take complex Al-Tememe transform $(\mathcal{T}^c)$ to both sides of (1), after simplification we can put $\mathcal{T}(y)$ as follows

$$\mathcal{T}^c(y) = \frac{h(ip)}{k(ip)}; \qquad h(ip) \neq 0 \tag{2}$$

where h, k are polynomials of ip such that the degree of h is less than the degree of k and the polynomial $.k$ with known prime cofactors. By taking $\mathcal{T}^{c-1}$ to both sides of equation (2) we will get:

$$y = \mathcal{T}^{c-1}\left[\frac{h(ip)}{k(ip)}\right]. \tag{3}$$

Equation (3) represents the general solution of the differential equation (1) which is form is given by :

$$y = A_0K_0(x) + A_1K_1(x) + \cdots + A_mK_m(x). \tag{4}$$

Such that $K_0, K_1, \cdots, K_m$ are functions of x and that $A_0, A_1, \cdots, A_m$ are constants, whose number equals to the degree of $k(ip)$.
To find the values of $A_0, A_1, \cdots, A_m$ we will substituting the initial conditions, one of them is $y(1)$ so we will get:

$$A_0K_0(1) + A_1K_1(1) + \cdots + A_mK_m(1) = y(1). \tag{5}$$

Take derivatives of (5) m times to get

$$A_0K_0'(1) + A_1K_1'(1) + \cdots + A_mK_m'(1) = y'(1) \tag{6}$$

$$A_0K_0''(1) + A_1K_1''(1) + \cdots + A_mK_m''(1) = y''(1) \tag{7}$$

$$\vdots$$

$$A_0K_0^{(m)}(1) + A_1K_1^{(m)}(1) + \cdots + A_mK_m^{(m)}(1) = y^{(m)}(1). \tag{8}$$

This is linear algebraic system can be solved to obtain $A_0, A_1, \cdots, A_m$ and so we obtain the solution of the required differential equation(1).
In equation (3), The polynomial $h(ip)$ is not necessary defined, it is indicated only by this symbol. While the polynomial (p) contains the multiplied $(a_0(ip)^n + a_1(ip)^{n-1} + \cdots + a_n)$ by the denominator of complex Al-Tememe transform for the function $f(x)$.
Note : We can put $\mathcal{T}^c(xy') = (ip-1)\mathcal{T}^c(y) + a$; a constant.

$$\begin{aligned}
\mathcal{T}^c(x^2y'') &= (ip-2)(ip-1)\mathcal{T}^c(y) + h_1(ip); \quad \text{degree of } h_1(ip) \text{ less than } 2 \\
\mathcal{T}^c(x^3y''') &= (ip-3)(ip-2)(ip-1)\mathcal{T}^c(y) + h_2(ip) \quad \text{degree of } h_2(ip) \text{ less than } 3 \\
&\vdots \\
\mathcal{T}^c(x^my^{(m)}) &= (ip-m)(ip-m+1)\cdots(ip-1)\mathcal{T}^c(y) + h_{m-1}(ip); \\
&\quad \text{degree of } h_{m-1}(ip) \text{ is less than } m.
\end{aligned}$$

Example 5 : To solve the differential equation $xy' - 2y = x^5; y(1) = 0$ we take complex Al-Tememe transform to both sides of above ODE we can write:

$$\mathcal{T}^c(y) = \frac{1}{(ip-3)(ip-6)}$$

so we write the general solution, after taking inverse of complex Al-Tememe transform, as follows:

$$y = Ax^2 + Bx^5. \tag{9}$$

Here equation (9) contains two constant A and B so we need two linear algebraic equations.
We get one of them by the initial condition $y(1) = 0$ so we get the equation:

$$A + B = 0. \tag{10}$$

For finding the second equation we should find additional condition which be get it from the above differential equation by substituting the initial condition $(1) = 0$ so we get :

$$y'(1) - 2y(1) = 1 \Rightarrow y'(1) = 1$$

and after taking derivative to equation (9) and substituted $y'(1) = 1$. We get :

$$2A + 5B = 1. \tag{11}$$

So, from (10) and (11) we get:

$$A = \frac{-1}{3}, \quad B = \frac{1}{3}$$

and hence the solution is given by:

$$y = \frac{-1}{3}x^2 + \frac{1}{3}x^5.$$

Example 6 : To solve the differential equation : $x^2y'' + 3xy' - 3y = x^{-2}lnx$; $y'(1) = y(1) = 0$. We take $\mathcal{T}$ to both sides of the differential equation we get:

$$\mathcal{T}^c(y) = \frac{1}{(ip-2)(ip+2)(ip+1)^2}. \tag{12}$$

After taking $\mathcal{T}^{c-1}$ to both sides of (12), we can write the general solution as follows:

$$y = Ax + Bx^{-3} + Cx^{-2} + Dx^{-2}lnx \tag{13}$$

where , A, B, C and D are constants.
To find the values of A, B, C and D we need four linear equations. We get the first equation by substituting $y(1) = 0$ in equation (13). For the second equation we derive the general solution (13) and substitute the initial condition $y'(1) = 0$. But the third and fourth equations we need two additional conditions. Now,

$$y''(1) + 3y'(1) - 3y(1) = 0 \Rightarrow y''(1) = 0$$
$$y'''(1) + 2y''(1) + 3y''(1) + 3y'(1) - 3y'(1) = 1 \Rightarrow y'''(1) = 1.$$

After substituting $y(1)=0, y'(1)=0, y''(1)=0, y'''(1)=1$ in y'', y''' respectively we get:

$$A+B+C=0, \tag{14}$$

$$A-3B-2C+D=0, \tag{15}$$

$$12B+6C-5D=0, \tag{16}$$

$$-60B-24C+26D=1. \tag{17}$$

So, from (14),(15), (16) and (17) we get :

$$A=\frac{1}{36},\quad B=\frac{-1}{4},\quad C=\frac{2}{9},\quad D=\frac{-1}{3}$$

so the solution of the differential equation takes the form:

$$y=\frac{1}{36}\ x-\frac{1}{4}x^{-3}+\frac{2}{9}x^{-2}-\frac{1}{3}x^{-2}lnx.$$

Solving Differential Equations with Multiple of Polynomial and Logarithms Coefficients by Using Al-Tememe Transform

Theorem 2 : If $(ip)=\mathcal{T}^c[f(x)]$ then

$$\mathcal{T}^c[(lnx)^k f(x))]=(-i)^k\frac{d^k}{dp^k}\mathcal{T}^c[f(x)];\quad k=1,2,3,\cdots$$

for a piecewise continuous function $f(x)$ on $[1,\infty)$.

Proof : Hence, for $k=1$

$$\mathcal{T}^c[ln\ x\cdot y]=-i\frac{d}{dp}\mathcal{T}^c(y).$$

Now

$$\begin{aligned}\mathcal{T}^c[ln\ x\cdot xy'] &= -i\frac{dP}{dp}\mathcal{T}^c(xy')\\ &= -i\frac{d}{dp}[-y(1)+(ip-1)\mathcal{T}^c(y)]\\ &= i[(ip-1)\frac{d}{dp}\mathcal{T}^c(y)+i\mathcal{T}^c(y)].\end{aligned}$$

$$\tau^c[ln\ \ \cdot xy']=(p+i)\frac{d}{dp}\mathcal{T}^c(y)+\tau^c(y).$$

Similarly, for

$$\begin{aligned}\mathcal{T}^c[ln\ x\ x^2y''] &= (-i)^2\frac{d}{dp}[-y'(1)-(ip-2)y(1)+(ip-1)(ip-2)\mathcal{T}^c(y)]\\ &= i^2[-iy(1)+(ip-2)(ip-1)\frac{d}{dp}\mathcal{T}(y)-(2p+3i)\mathcal{T}(y)]\end{aligned}$$

$$\mathcal{T}^c[ln\ x\ x^2y''] = iy(1)-(ip-2)(ip-1)\frac{d}{dp}\mathcal{T}^c(y)-i(2ip+3)\mathcal{T}^c(y).$$

In many cases these formulas are useful to solve linear differential equations with variable coefficients.

Example 7 : To solve the differential equation: $xlnx \cdot y' - y = (lnx)^3$.

Solution : By taking $(\mathcal{T}^c)$ to both sides we get:

$$\mathcal{T}^c[xln\ x \cdot y'] - \mathcal{T}^c(y) = \mathcal{T}^c[(lnx)^3]$$

$$(p+i)\frac{d}{dp}\mathcal{T}^c(y)+\mathcal{T}^c(y)+\mathcal{T}^c(y) = \frac{3!(-ip-1)}{(p^2+1)^4}$$

$$\frac{d}{do}\mathcal{T}^c(y)+\frac{2}{p+i}\mathcal{T}^c(y) = \frac{3!(-ip-1)}{(p^2+1)^4(p+i)}.$$

This is linear ordinary differential equation and its integrating factor is given by:

$$\mu = e^{\int \frac{2}{p+i}dp} = e^{2ln}(p+i) = (p+i)^2.$$

Therefore,

$$\frac{d}{dp}[\mathcal{T}^c(y)\cdot(p+i)^2] = \frac{3!(ip+1)^4(p+i)^2}{(p^2+1)^4(p+i)} = 3!(p+i)^{-3}$$

$$\begin{aligned}\mathcal{T}^c(y)\cdot(p+i)^2 &= \int 3!(p+i)^{-3}dp \to \mathcal{T}^c(y)\cdot(p+i)^2\\ &= \frac{3!}{-2(p+i)^2}\\ &= \mathcal{T}^{c^{-1}}\frac{-3(p-i)^4}{(p^2-1)^4}+\frac{c(p-i)^2}{(p^2-1)^2} = -3(lnx)^3 + c\ lnx.\end{aligned}$$

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