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# GENERALIZED SPECIAL FUNCTIONS AND POLYNOMIALS

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**Abstract**

In this paper our aim is to study the generating relation of generalized special functions and to obtain the rational and polynomial approximation of the above function. It is also proposed to find the integral transform of the above function and deduce some properties. It is also proposed to study certain class of generalized polynomials represented by a generalized Rodrigues formula and to derive their linear generating relations. It may be expanded in terms of generalized heat Polynomials.

## 1. Introduction

### 1.1 Rodrigue's Formulae and Generalizations due to it :

A formula, which connects a polynomial to the $n-th$ derivative of a function, is called a Rodrigue's formula for Exp.

$$H_n(x) = (-1)^n \exp(x^2)\frac{d^n}{dx^n}[\exp(-x^2)], \quad \text{(Hermite)} \tag{1.1.1}$$

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{\underline{|n}}\frac{d^n}{dx^n}(e^{-n}x^{n+\alpha}), \qquad \text{(Laguerre)} \tag{1.1.2}$$

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Key Words : *Certain classical generalized polynomials, Rodrigue's formula and generalization due to it, Generalized with the help of differential equations.*

$$P_n(x) = \frac{1}{2^n \underline{|n}} \frac{d^n}{dx^n}(x^2-1)^n, \qquad \text{(Legendre)} \tag{1.1.3}$$

and

$$P_n^{a,b} = \frac{(-1)^n(1-x)^{-a}(1+x)^{-b}}{2^n \underline{|n}} \frac{d^n}{dx^n}[(1-x)^{n+a}(1+x)^{n+b}], \qquad \text{(Jacobi)} \tag{1.1.4}$$

These formula have been the sources of numerous researches.

(i) Another generalization due to Nicolas [6] is

$$P_n[x, f(x)] = \frac{1}{k^n \underline{|n} \underline{|(k-1)n}} \frac{d^{(k-1)n}}{dx^{(k-1)n}}[f(x)^n; \tag{1.1.5}$$

Where $k$ is an integar $\geq 2$ and $f(x)$ is a polynomial of degree $k$.

When $K = 2$ and $f(x) = (x^2-1)$ in (1.1.5), $P_n[x, f(x)]$ transforms into the Legendre Polynomials.

(ii) In 1929, Ghosh, N.N., has generalized the polynomials defined in (1.1.5) by the expression

$$\lambda_n a, b = \frac{d^n}{dx^n}\left[x^a\left(\frac{1}{x}-x\right)^b\right] \tag{1.1.6}$$

where $a, b$ are arbitrary constants. (1.1.6) evidently includes Legendre Polynomials too, for we have

$$\lambda_{n,n,n} = (-1)^n 2^n \underline{|n} p_n(x).$$

(iii) Nicolas, Cioransen [6] has obtained a generalization of Legendre polynomials by defining them by the formula

$$P_n^{(x;Q)} = \frac{1}{A_n} \frac{d^{(k-1)n}}{dx^{(k-1)n}}[\{Q(x)\}^n] \tag{1.1.7}$$

where $Q(x) = (x-a_1)(x-a_2), \cdots (x-a_k)$ is a Polynomial of degree $k$ and $A_n$ is a suitable constant.

(iv) B. S. Sastry has given the generalization of Laguerre Polynomials in another form, viz.,

$$\pi_n(x) = e^x \frac{d^n}{dx^n}[e^{-x}\overline{A}_n(x)]; \tag{1.1.8}$$

where

$$\overline{A}_n(x) = (a_0, a_1, a_2, \cdots, a_n x, 1)^n.$$

(v) Kesava Menon [8] has generalized $P_n(x)$ in the form

$$P_{n;s}(x) = \frac{1}{s^n \underline{|n}} \frac{d^n}{dx^n}(x^s - 1)^n \tag{1.1.9}$$

where $s$ is an integer $\geq 2$.

When $s = 2$ the Polynomial $P_{n,s}(x)$ becomes Legendre Polynomial. He has also derived some properties of $P_{n,s}(x)$.

(vi) The generalization due to A. Angelscu [4] is

$$\pi_n(x) = \frac{e^{-x}}{\underline{|n}} \frac{d^n}{dx^n}[e^x \cdot A_n(x)] \tag{1.1.10}$$

where $A_n(x)$ is polynomial of degree $n$ in $x$, such that

$$\frac{d}{dx}A_n(x) = nA_{n-1}(x).$$

(vii) Maurice de Duffabel has gone through the polynomials. $P_n(x)$ where

$$P_n(x) = \frac{e^{x^2}}{\underline{|n}} \frac{d^n}{dx^n}(x^n \cdot e^{-x^2}). \tag{1.1.11}$$

(viii) In (1901), Appell [2] has considered the class of polynomials

$$R_{2n}(x) = D^n\{x^n(1 - x^2)^n\}. \tag{1.1.12}$$

(xi) Starting with a Rodrigue's formula, another interesting study is due to E. T. Bell. He starts with the study of the Polynomials.

$\xi_n(x, t, r)$ given by the relation

$$\xi_n(x, t, r) = \exp(-r, t^r)\frac{d^n}{dx^n}(\exp \cdot (xt^r)) \tag{1.1.13}$$

(x) In (1955), P. C. Chetterjee has generalized the ordinary Hermite Polynomials $H_n(z)$ of integral order $n$. The generalized Hermite Polynomials. $H_{km}(z)$ and $H_{km+1}(z)$ are defined as

$$H_{km}(z) = e^{zk}D_{(k)}^{(km)}(e^{-z^k}) \tag{1.1.14}$$

$$H_{km+1}(z) = -e^{zk}d_{(k)}^{(km+k-1)}(e^{-z^k}) \tag{1.1.15}$$

where

(i) $m$ and $mk$ are non-negative integers

(ii) $k \neq 0$ (clearly $k$ may also be fractional)

(iii) $D_{(k)}^{'(k)} = \frac{d}{dz} \frac{1}{z^{k-2}} \frac{d}{dz}$

(iv) $D_{(k)}^{'(k-1)} = \frac{d}{dz} D_{(k)}^{(k)}$ so that

(v) $D_{(k)}^{'(k-1)} = \frac{1}{z^{k-2}} \frac{d}{dz}$ and

(vi) $D_{(i)}^{'(km)}$ means the operator

$D_{(k)}^{'(km)}$ operating on a function m times successively.

(xi) The generalization due to Subba Rao, is

$$R_n[x, a, b; f(x)] = \frac{1}{A(n)} \frac{d^{na}}{dx^{na}} [f(x)]^n \tag{1.1.16}$$

where $f(x)$ is a polynomial of degree $n$, $b > a \geq 1$, $a, b$ being +ve integers and $A(n)$ a constant; which is a suitable function of $a, b$ and $n$ only.

(xii) Let $D_k^k$ stands for the operator $\frac{d}{dz}\left(\frac{1}{z^{k-2}} \cdot \frac{d}{dz}\right)$ and $D_k^{km}$ for the operator $(D_k^k)$ repeated $m$ times. In (1948), Sharm [7] has shown that

$$\begin{aligned} 2F_i\left[-m, m + \tfrac{1}{k}; 1(1 - z^k)^{2m}\right] &= Q_{km}(z) \\ &= \tfrac{1}{k^{2m}\underline{|2m}} D_k^{km}(1 - z^k)^{2m} \end{aligned} \tag{1.1.17}$$

where $k = 2$, it reduces to

$$P_{2m}(z) = \frac{1}{2^{2m}\underline{|2m}} \frac{d^{2m}}{dz^{2m}} (1 - z^2)^{2m} \tag{1.1.18}$$

which is the well-known Rodrigue's formula for the Legendre Polynomials.

(xiii) Chatterjee, S. K. [4] has shown that if $k$ be a positive integer than

$$\frac{1}{\underline{|n}} x^{-\alpha} e^{px^k} \frac{d^n}{dx^n} (x^{\alpha+n} e^{px^k})$$

is a polynomial of degree $n$ and is denoted by

$$T_{k,n}^{(\alpha)}(x, p) \tag{1.1.19}$$

$$T_n^{(\alpha)}(x, 1) = L_n^{(\alpha)}(x).$$

He has defied an operational formula, a generating function and various recurrence formulae, which are generalizations of the corresponding formulae for Lagurre polynomials.

**1.2 Generalization of the Classical Polynomials with the help of Hypergeometric forms**

In (1947), Mery celine Fasenmyer [5] has obtained some basic formal properties of the herpergeometric polynomials.

$$\begin{aligned} f_n(a_i;b_j;x) &= f_n(a_1;a_2;\cdots;a_p,b_1,b_2,\cdots,b_q(x)) \\ &= {}_{p+2}F_{q+2}\left[\begin{array}{cccc} -n & n+1, & (a_p) & \\ & & & ;z \\ 1/2, & 1, & (b_q) & \end{array}\right] \end{aligned} \tag{1.2.1}$$

($n$; a non-negative integer), in an attempt to unify and to extend the study of certain polynomial sets.

**SOME SPECIAL CASES OF $f_n(a_i;b_j;x)$ are**

(i) $f_n(1/2;-;x) = P_n(1-2x)$ (Legendre)

(ii) $f_n(1;-;x) = \left\{\frac{|n}{(1/2)_n}\right\} P_n^{(-1/2,-1/2)}(1-2x)$ (Jacobi)

(iii) $f_n(1;1/2;b,x) = \left\{\frac{|n}{(b)_n}\right\} P_n^{(b-1,1-b)}(1-2x)$ (Jacobi)

(iv) $f_n(1/2;a;p;v) = H_n(a,p,v)$ (Rice's)

(v) $f_n\left(\frac{1}{2};\frac{1+z}{2};1;1\right) = F_n(z)$ (Bateman's)

(vi) $f_n(1/2;1;t) = Z_n(t)$ (Bateman's)

(vii) $f_n\left(\frac{1}{2},\frac{z+m+1}{2};m+1;1\right) = F_n^m(z).$ (Pasternak's)

A generating function, differential and pure recurrence relations, contiguous polynomial relations and integral relations for the polynomial $f_n(a_i;b_j;x)$ have also been given by her.

2. A natural generalization of Laguerre Polynomial in the form of a constant multiple of Certain, ${}_1F_q$, viz.

$$ {}_1F_q\left[-n;\frac{\alpha+1}{q};\cdots\frac{\alpha+q}{q};\left(\frac{x}{q}\right)^q\right] \quad (q=0,1,2,\cdots) \tag{1.2.2}$$

Has been given by Tascano [8] in (1956).

3. Palam G. in (1950), has generalized $L_n^{(a)}(x)$ in the form

$$L_n^{(a)}(x) = \frac{(1+\alpha)_n}{\underline{|n}} \psi_1[-n, v+1; -\alpha-2n-1/2+1; 2x] \tag{1.2.3}$$

where $\psi_1$ is one of the Humbert's confluent hypergeometric functions of two variables.

### 1.3 Generalizations due to Series Representations

Struve's function has been defined by the series

$$H_v(z) = \sum_{r=0}^{x} \frac{(-1)^r \left(\frac{z}{2}\right)^{v+2r+1}}{\left|\overline{\left(r+\frac{3}{2}\right)} \left(\left|\overline{v+r+\frac{3}{2}}\right.\right)\right.}. \tag{1.3.1}$$

H. C. Gupta has defined the Struve's function by means of the relation

$$H_v(z) = \frac{2\left(\frac{z}{2}\right)^{v+1}}{\left|\overline{\left(\frac{1}{2}\right)} \left(\left|\overline{v+\frac{3}{2}}\right.\right)\right.} \phi\left(1; 3/2, v+3, \frac{1}{4}z^2\right). \tag{1.3.2}$$

Another generalization of Struve's function has been contributed by Bhowmick, K. N. [3] in the form

$$\mu H_v(z) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{z}{2}\right)^{v+2r+1}}{\left|\overline{\left(r+\frac{3}{2}\right)} \left(\left|\overline{v+r+\frac{3}{2}}\right.\right)\right.}, \quad (\mu > 0) \tag{1.3.3}$$

If $\mu = 1$; this reduces to the ordinary struves function.

In 1963, Ranyaranjan, S. K. has generalized the Laguerre Polynomials as follows:

$$\frac{\pi_n^{(\alpha)} x}{(1+\alpha)} = \sum_{m=0}^{\infty} \begin{pmatrix} n \\ m \end{pmatrix} \frac{(-1)^m}{(1+\alpha)_n} \overline{A_m}(x) \tag{1.3.4}$$

where

$$\overline{A_n}(x) = a_0 x^m + \begin{pmatrix} m \\ 1 \end{pmatrix} a_1 x^{m-1} + \cdots + a_{m-1} \begin{pmatrix} m \\ m-1 \end{pmatrix} x + a_m.$$

If $\alpha = 0$, he found (1.1.8).

He has also derived relation between $\pi_n^{(\alpha)}(x)$ and $L_n^{(\alpha)}(x)$. Some finite series involving $\pi_n^{(\alpha)}(x)$, relations between $\pi_n^{(\alpha)}(x)$ and Bernoulli polynomials, relation between $\pi_n^{(\alpha)}(x)$ and Legendre polynomials and integrals involving $\pi_n^{(\alpha)}(x)$.

### 1.4 Generalization with the help of Differential Equations

Menon, P. K. has generalized the Legendre functions by generating the legendre differential equations viz.

$$(1-z)^2\frac{d^2y}{dz^2}-2z\frac{dy}{dz}+n(n+1)y=0 \tag{1.4.1}$$

This equation may be identified with hypergeometric equation.

$$\left[z^2\frac{d}{dz^2}\left(z^2\frac{d}{dz}-\frac{1}{2}\right)-z^2\left(z^2\frac{d}{dz^2}-\frac{n}{2}\right)\left(z^2\frac{d}{dz^2}+\frac{d+1}{2}\right)\right]y=0 \tag{1.4.2}$$

where $n$ is a positive integer, the differential equation admits of a polynomial solution and a non-polynomials solution, both of which have well-known properties:
He has considered the generalized Hypergeometric equation.

$$\left\{\begin{array}{l} w\left(w-\frac{1}{s}\right)\left(w-\frac{2}{s}\right)\cdots\left(w-\frac{s-1}{s}\right)-z^s\left(w-\frac{n(s-1)}{s}\right) \\ \times\left(w+\frac{n+1}{s}\right)\left(w+\frac{n+2}{s}\right)\cdots\left(w+\frac{n+s-1}{s}\right) \end{array}\right\}y=0 \tag{1.4.3}$$

where $w=z^2\frac{d}{dz^s}$.
When $n$ is a positive integer, this has one polynomial solution and $(s-1)$ other solutions.

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