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ON THE SINGULARITY OF INFINITE DIMENSIONAL MANFOLDS

FAWAZ ALHARBI[1] AND ELSAEED AL-AZAB[2]
[1,2] Debartment of Mathematical Sciences,
Umm Alqura University, Makkah, P.O.Box:715, Saudi Arabia

Abstract

The singularity of the coordinates in the case of infinite dimensional manifolds Is discussed has no sense. Also, Gauss and Codazzi equations for hypersurfaces in a Banash manifold are established. A generalization of the Schur theorem to the case of Banach manifolds is given. Concepts of bending and equiaffinity are introduced for infinite dimensional hypersurfaces in Hilbert manifolds and theorem on local isometry of Hilbert manifolds of same constant sectional curvature is proved. Finally, the class of hypersurfaces equiaffine to the hypersphere is described.

1. Notation and Definitions

By singularity, we mean a point at which a given mathematical object is not defined or not "well behaved". In infinite dimensional manifolds, the coordinates has no sense. Therefore, to avoid such singularity we shall consider finite codimensional Banach submanifolds.

Key Words : *Singularity, Banach manifold, Gauss and Codazzi equations.*
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Let M be a Banach manifold of class C^r $(r \geq 2m\infty)$ modeled on a Banach space E and let N.. be a submanifold of M of the same class [3]. By Di_x we denote the Frechet derivative of the inclusion map $i : N \to M$ at the point $x \in N$ and T_xM the space of all tangent vectors of M at the point $x \in N$ [2].

Let $(M, \overline{G})$ be a Riemannian manifold and N a submanifold of M with induced metric $\overline{g}$. The symmetric bilinear positive definite continuous functional $f \in L_2(E; R)$ is said to be strongly non-singular if f associates a mapping

$$f^* : x \in E \to f^*_x = f(x, \cdot) \in L(E; R) = E^*$$

which is bijective [2].

Assume that the metrics $\overline{G}$ and $\overline{g}$ are positive definite and strong non-singular. By codim $N = k < \infty$, we mean that there exist charts $C = (U, \phi, E)$ at the point $x \in M$ and $D = (V = U \cap N, \psi = \O/V, F \subset E)$ at $x \in N$ such that

$$codim\ F = dim(E/F) = k < \infty.$$

Assume that the chart C is fixed at $x \in M$. Define a mapping $\omega_{C,x} : T_xM \to E$ as follws: Let $\overline{h} \in T_xM$. From all equivalence pairs which define the vector $\overline{h}$, we take the pair (C, h) whose first componenet is our fixed chart C, then the second component h can be taken as the image of $\omega_{C,x}$ at $\overline{h}$.

Let $L_{m_1+m_2}(T_xM, T_xN; R)$ (resp., $L_{m_1+m_2}(T_xM, T_xN; T_xM)$ and $L_{m_1+m_2}(T_xM, T_xN; T_xN)$) be the space of all $(m_1 + m_2)$-linear continuous functionals (resp., mappings) from $(T_x, M)^{m_1} \times (T_xN)^{m_2}$ into R (resp., into T_xM and T_xN) which is complete normed space [3]. Vectors of these spaces are called mixed tensors of type $(0+0, m_1+m_2), (1+0, m_1, +m_2)$ and $(0+1, m_1+m_2)$ resp. at the point $x \in N \subset M$ where $m_1+m_2 \geq 0$.

We shall denote the set of all tensors of the type $(S_1 + S_2, m_1 + m_2)$ at the point $x \in N \subset M$ by $T^{s_1+s_2}_{m_1+m_2}(x)$ where $s_1 + s_2 = 0, 1$.

By $\nabla^{1,2}$ we denote the operation of mixed covariant differentiation [4]. If $\overline{B} \in T^{1+0}_{m_1+m_2}(N)$, then $\nabla^{1.2}\overline{B} \in T^{1+0}_{m_1+(m_2+1)}(N)$ and if $C = (Uk, \phi, E)$ and $D = (V, \psi, F)$ are two charts at the point $x \in N \subset M$ and $\overline{\Gamma}, \overline{\gamma}$ are the induced linear connections on M, N respectively, then for all $P = \psi(x) \in \psi(V) \subset F, y_1, \cdots, y_{m_1} \in E$ and $h, h_1, \cdots, h_{m_2} \in F$

$$\nabla^{1,2} B_p(h; y_1, \cdots, y - m_1, h_1, \cdots, h_{m_2}) = B_p(h; y_1, \cdots, y_{m_1}, h_1, \cdots h_{m_2})$$
$$-\sum_1^{m_1} B_p(y_1, \cdots, y_{j-1}, \Gamma_{i(p)}(y_j, D_{i_p}(h)), , y_{j+1}, \cdots, y_{m_1}, h_1, \cdots, y_{m_2})$$
$$-\sum_1^{m_2} B_p(y_1, = \cdots, y_{m_1}, h_1, \cdots, h_{j-1}, \gamma_p(h_j, h), h_{j+1}, \cdots, h_{m_2})$$
$$+\Gamma_{i(p)}(B_p(y_1, \cdots, y_{m_1}, h_1, \cdots, h_{m_2})), D_{i_p}(h)).$$

2. Auxiliary Assertions

Assuming that at every point $x \in N$, the tangent space T_xN to the submanifold $N \subset M$ has an orthogonal complement $(T_x, N)^\perp$,

$$(T_xN)^\perp = \{\overline{Y} \in T_xN : \overline{G}_x(\overline{Y}, \overline{X}) = 0 \text{ for all } \overline{X} \in T_xN\}$$

Such that $T_xN \oplus (T_xN)^\perp = T_xM$ and the Banach spaces $T_xN \times (T_xN)^\perp$ and T_xM are isomorphic (here $\oplus$ is the operation of the direct sum of mutually orthogonal subspaces T_xN and $(T_xN)^\perp$ [3]. From the definition of the submanifold there exist charts $C = (U, \phi, E)$ on M atthe point $x \in M$ and $D = (V = U \cap N, \psi = \emptyset/N, kF \subset E)$ on N at the point $x \in N$ such that $\emptyset(V) \subset F$.

Then $\omega_{C,x}((T_xN)^\perp) = F^\perp_{p=\psi(x)} \subset E$, where $F^\perp_p$ is the orthogonal complement of $F \subset E$ with respect to the metric G, i.e.m for all $Z \in F$.

$$G_{i(p)}(Z, D_{ip}(X)) = 0. \tag{1}$$

In [4] it is proved that all orthogonal comlements $F^\perp_p$ of F are isomorphic to a Banach space W and for every $x \in N \subset M$ there exists an isomorphism

$$\overline{n}_x : W \to (T_xN)^\perp \subset T_xM$$

which satisfies the following property: For all x in N, there exist charts $D = ()V, \psi, F)$ at $x \in N$ and $C = (U, \emptyset, E)$ at the point $\bar{\iota}(x) = x$ on M such that the mapping

$$n = p = \psi(x) \in \psi(V) \subset F \to n_p = \omega_{C,x} \ o \ \overline{n}_x \in L(W; F^\perp_p)$$

is of class c^{r-1}.

Also, in [4] the first and the second derivative equations of the submanifold N are established in the forms:

For all $p = \psi(x) \in \psi(V) \subset F$ and for all $X \in F, Z \in W$,

$$\nabla^{12} D_{i_p}(X_1, X_2) = n_p(A_p(X_1, X_2)) \tag{2}$$

$$D_{n_p}(X, Z) = D_{i_p} H_p((X, Z)) + n_p(S_p(X, Z)) \tag{3}$$

where$A_p \in L_2(F; W), H_p \in L(F, W; F)$ and $S_p \in L(F, W; W)$ are class C^{r-2} and n_p is the model of $\overline{n}_p$ with respect to the chart C.

3. Gauss and Codazzi Equations Gauss and Codazzi Equations for a Hypersurface in a Banach Manifold

Now from integrability condition for the Equations (2),(3), Gauss and Codazzi equations for the submanifold N in the Banach manifold M are obtained in [4] which have the forms:

$$\begin{aligned}
&R_{i(p)}(D_{i_p}(X_2); D_{i_p}(X_3), D_{i_p}(X_1)) + D_{i_p}((H_p(\underline{X}_3, A_p(\underline{X}_1, X_2)) \\
&+n_p(S_p(\underline{X}_3, A_p(\underline{X}_1, X_2)) + DA_p(\underline{X}_3; \underline{X}_1, X_2) - A_p(\underline{X}_1, \gamma_p(X_2, \underline{X}_3))) \\
&+\Gamma_{i(p)}(n_p(A_p(\underline{X}_1, X_2)), D_{i_p}(\underline{X}_3)) = D_{i_p}(r_p(\underline{X}_2, X_3, \underline{X}_1)) \\
&+D_p(DH(\underline{X}_1; \underline{X}_2, X_3) + \gamma_p(H_p(\underline{X}_2, X_3), \underline{X}_1) + H_p(\underline{X}_1, S_p(\underline{X}_2, X_3))) \\
&+n_p(DS_p(\underline{X}_1; \underline{X}_2, X_3) + S_p(\underline{X}_1, S_p(\underline{X}_2, X_3)) \\
&+A_p(\underline{X}_1, H_p(\underline{X}_2, X_3))) + \Gamma_{i(p)}(D_{i_p}(H_p(\underline{X}_2, X_3)), , D_{i_p}(\underline{X}_1)) = 0
\end{aligned}$$

where R and r denote the models of the curvature tensors $\overline{R}$ and $\overline{r}$ on the Banach manifolds M and N with respect to the charts C on M and D on N respectively. Similarly, $\overline{\Gamma}$ and $\overline{\gamma}$ are the models of the free-torsion connection Γ and γ on M and N with respect to the charts C and D respectively.

Remark : In Gauss and Codazzi equations there exists an alternation with respect to the underlined vectors, that does not involve division by

2. This convention will be used henceforth.

Gauss and Codazzi equations make more precise in the case of finite codimensional Banash submanifolds. For this purpose let codim $N = k < \infty$. This means that there exist a chart $C = (U, \phi, E)$ at the point x on M and a chart $D = (V = \cap N, \psi = \emptyset/v, F \subset E)$ at x on M such that codim $F = k < \infty$.

Now we consider the following theorem [las2] :

Theorem 3.1 : Let E be a Banach space, $F \subset E$ be a closed subspace of E such that codim $F = k < \infty$ and let G be a strongly non-singular, symmetric bilinear form on E such that G/F is weekly non-singular (i.e. $G(Z, Y) = 0$ for all $Y \in F$ implies $Z = 0$). Then there exists k-dimensional vector subspace W orthogonal complement to F with respect to G such that G/w is weekly non-singular form. Therefore for every $Z \in W$ we have that $Z = \sum_1^k z^n e_n$ where e_n are fixed non-zero independent vectors in W. Now, let $n_p(e_n) = \xi_{pn} \in (T_xN)^\perp$ is a differentiable vector of class C^{r-1} for all $n = 1, \cdots, k$. Hence without loss of generality we can choose $e_n \in W$ such that the vectors ξ_{pn} are orthonormal for all $n = 1, \cdots, k$. Therefore

$$G_{i(p)}(\xi_{pn}, D_{i_p}(X)) = 0 \;\text{ for all }\; n = 1, \cdots, k. \tag{4}$$

Similarly, denoting $H_p(X, e_n) = H_{pn}(X) \in F$, then we have:

$$A_p(X_1, X_2) = \sum_1^k A_{pn}(X_1, nW \tag{5}$$

$$S_p(X, e_n) = S_{pjn} \sum_1^k S_{pnm}(X) e_m \in W. \tag{6}$$

If we consider Equations (5),(6) then (2),(3) take the forms

$$\nabla^{12} D_{i_p}(X_1, X_2) = \sum_1^k A_{pn}(X_1, X_2)\xi_n \tag{7}$$

$$D\xi_{p_n}(X_1) = D_{i_p}(H_{pn}(X_1)) + \sum_{m=1}^k S_{pnm}(X_1)\xi_{nm}. \tag{8}$$

Applying (5), (6), in Gauss equation and multiplying it's both sides by $Di_p(X_3)$ with respect to the metric G and using each of (1), (4) we obtain

$$G_{i(p)}(R_{i(p)}(D_{i_p}(X_2); D_{i_p}(X), D_{i_p}(X_1)), D_{i_p}(X_3))$$
$$+ \sum_1^k A_{pn}(\underline{X}_1, X_2) G_{i(p)}(D_{i_p}(H_{pn}(\underline{X})), D_{i_p}(X_3))$$

$$+\sum_{1}^{k} A_{pn}(\underline{X}_1, X_2)G_{i(p)}(\Gamma_{i_p}(\underline{X}), D_{i_p}(X_3)) = g_p(r_p(X_2, X, X_1), X_3). \tag{9}$$

Now from (2) we can deduce that

$$\nabla^{12}\xi_{pn}(X) = D\xi_{pn}(X) + (\Gamma_{i(p)}(\xi_{pn}, D_{i_p}(X)) \tag{10}$$

$$G_{i(p)}(\nabla^{12}\xi_{pn}(X), D_{i_p}(X_3)) = -G_{i(p)}(\xi_{pn}, \nabla^{12}D_{i_p}(X, X_3)). \tag{11}$$

Then substituting (8) into (9) and using (4), (10) and (11) we get

$$G_{i(p)}(R_{i(p)}(D_{i_p}(X_2); D_{i_p}(X), D_{i_p}(X_1)), D_{i_p}(X_3))$$

$$+\sum_{1}^{k} A_{pn}(\underline{X}, X_2)G_{i(p)}(\xi_{pn}\nabla^{12}D_{i_p}(\underline{X}_1, X_3) = g_p(r_p(X_2; X, X_1), X_3)). \tag{12}$$

substituting (7) into (12) we have

$$G_{i(p)}(R_{i(p)}(D_{i_p}(X_2); D_{i_p}(X), D_{i_p}(X_1)), D_{i_p}(X_3))$$

$$+\sum_{1}^{k} A_{pn}(\underline{X}, X_2)A_{pn}(\underline{X}_1, X_3) = g_p(r_p(X_2; X, X_1), X_3). \tag{13}$$

This is called Gauss equation for a Banach submanifold N of finite codimension k in a Banach manifold M.

Applying (5),(6) in Gauss equation, then multiplying the result by ξ_{pm} with respect to the metric G and using (4) we obtain

$$G_{i(p)}(R_{i(p)}(D_{i_p}(X_2); D_{i_p}(X_3), D_{i_p}(X_1)), \xi_{pn})$$

$$+\sum_{1}^{k} A_{pm}(\underline{X}_1, X_2)S_{pnm}(\underline{X}_3) + \nabla^{12}A_{pm}(\underline{X}_3, \underline{X}_1, X_2)$$

$$+\sum_{1}^{k} A_{pm}(\underline{X}_1, X_2)G_{ip}(\nabla^{12}\xi_{pm}(\underline{X}_3)) - D\xi_{pm}(X), D\xi_{pm} = 0. \tag{14}$$

Substituting by (8) into (14) and taking (4) into account we have that

$$G_{i(p)}(R_{i(p)}(D_{i_p}(X_2); D_{i_p}(X_3), D_{i_p}(X_1)), \xi_{pn})$$

$$+\sum_{1}^{k} A_{pm}(\underline{X}_1, X_2)G_{ip}(\nabla^{12}\xi_{pm}(\underline{X}_3), \xi_{pn}) = (\nabla^{12}A_{pm}(\underline{X}_1; \underline{X}_3, X_2) \tag{15}$$

Which is called Codazzi equation for a Banach submanifold N of finite codimension k in a Banach manifold M. Note that at $k = 1$ we obtain the equations of a hypersurface N in a Banach manifold M.

4. Banach Manifold of Constant Sectional Curvature

Suppose that M be a differentiable manifold of constant sectional curvature [2] of class C^r $(r \geq 3)$. Then the curvature tensor on M has the form [2]: For all $x \in M, \overline{X}_1, \overline{X}_2, \overline{X}_3 \in T_x M$,

$$\overline{R}_x(\overline{X}_3, \overline{X}_1, \overline{X}_2) = \beta_x(\overline{G}_3(\overline{X}_3, \overline{X}_2) \cdot \overline{X}_1 - \overline{G}_x(\overline{X}_3, \overline{X}_1), \overline{X}_2) \tag{16}$$

where β_x is a real function of points of M. Now substituting Equation (1) and (16) into (13) we obtain:

$$g_p(\beta_p g_p(X_2, \underline{X}_1) \cdot \underline{X} - r_p(X_2; X, X_1), X_3) = \sum_1^k A_{pm}(\underline{X}, X_2) A_{pn}(\underline{X}_1, X_3) = 0 \quad 17)$$

which is called Gauss equation for a Banach submanifold N of finite codimension k in a differentiable manifold M of constant sectional curvature β.

Theorem 4.1 : If at each point of M the equality (17) holds, then β is constant.

Proof : Let M be a Riemannian manifold of constant sectional curvature β modeled on a Banach space E. It is sufficient to prove the theorem locally with respect to an arbitrary chart $C = (U, \phi, E)$ at a point $x \in M$. Let $C' = (U', \phi', E')$ be another chart at the point $x \in M$. It is easy to see that β is independent of the choice of the choice of the chart [las1]. Hence the transformation from the chart C to the chart C' does not change the mapping β_p, therefore

$$\nabla \beta_p = D\beta_p. \tag{18}$$

Covariant differentiation of (16) locally with respect to $X_3 \in E$ yields.

$$\nabla R_p(X_4; X_3, X_1, X_2) = D\beta_p(X_4) G_p(X_3, \underline{X}_2) \cdot \underline{X}_1, \tag{19}$$

then substituting (19) in Bianchi's identity [las3] we get

$$D\beta_p(\underline{X}_1) G_p(X_2, \underline{X}_4) \cdot X_3 + D\beta_p(X_3) G_p(X_2, \underline{X}_1) \cdot \underline{X}_4 + D\beta_p(\underline{X}_4) G_p(X_2, \underline{X}_3) \cdot X_1 = 0.$$

Seen X_1, X_2, X_3 are arbitrary vectors in E, then we can take $D\beta_p(\underline{X}_1)G_p(X_2, \underline{X}_4) = 0$. But G_p is non-singular and X_2 is arbitrary in E, then we have $D\beta_p(\underline{X}_1), \underline{X}_4 = 0$. Also, since X_1, X_4 are arbitrary in E, then $D\beta_p = 0$ for all $p \in E$ which implies that β is constant.

This theorem generalizes in the infinite dimensional case, the Schur theorem which is known for finite dimensional manifolds [eize].

Hilbert Manifolds

Now, if M is a Hilbert manifold, then the metric G will be the scalar product $\langle \cdot, \cdot \rangle$ on E and the linear connection Γ on M will vanish, consequently the curvature tensor R on M will be identically zero. Then Gauss equation will be

$$D_{i_p}(H_p(X_3, A_p(\underline{X}_1, X_2))) + n_p(S_p(\underline{X}_3, A_p(\underline{X}_1, X_2))$$

$$+DA_p(\underline{X}_3; \underline{X}_1, X_2) - A_p(\underline{X}_1, \gamma_p(X_2, \underline{X}_3))) = D_{i_p}(r_p(X_2; X_3, X_1)). \tag{20}$$

Also, in this case Equations (1) and (2) will take the forms

$$g_p(X_1, X_2) = \langle D_{i_p}(X_1), D_{i_p}(X_2)\rangle, \tag{21}$$

$$\langle n_p(Z), D_{i_p}(X)\rangle = 0, \tag{22}$$

for all $X_1, X_1 \in F$ and $Z \in W$.

Scalar multiplication of both sides of (20) by $D_{i_p}(X)$ and using (21), (22) gives us

$$g_p(H_p(X_3, A_p(\underline{X}_1, X_2)), X) = g_p(r_p(X_2; X_3, X_1), X).$$

Since g_p is non-singular, hence

$$r_p(X_2; X_3, X_1) = -H_p(\underline{X}_1, A_p(\underline{X}_3, X_2)), \tag{23}$$

which is called Gauss equation for a submanifold N of a Hilbert manifold M.

A second scalar multiplication of both sides of (20) by $Z \in W$ and using that Z is arbitrary and n_p is injective yields

$$A_p(\underline{X}_1, \gamma_p(X_2, \underline{X}_3)) + DA_p(\underline{X}_3, \underline{X}_1, X_2) = S_p(\underline{X}_1, A_p(\underline{X}_3, X_2)). \tag{24}$$

Similary, since M is a Hilbert manifold, then Codazzi equation takes the form:

$$D_{i_p}(DH_p(\underline{X}_1, \underline{X}_2, Z) + \gamma_p(H_p(\underline{X}_2, Z)\underline{X}_1) + H_p(\underline{X}_1, S_p(\underline{X}_2, Z)))$$

$$+n_p(DS_p(\underline{X}_1; X_2, Z) + S_p(\underline{X}_1, S_p(\underline{X}_2, Z) + A_p(\underline{X}_1, H_p(\underline{X}_2, Z))) = 0. \qquad (25)$$

Scalar multiplication of (25) by $D_{i_p}(X_3)$, using (21), (22), taking into account that g_p is non-singular and X_3 is arbitrary we obtain

$$DH_p(\underline{X}_1; \underline{X}_2; Z) = \gamma_p(H_p(\underline{X}_2, Z), X_1) + H_p(\underline{X}_1, S_p(\underline{X}_2, Z)) = 0. \qquad (26)$$

A second scalar multiplication of (25) by $Z \in W$, using (22), taking into account that Z is an arbitrary and n_p is injective mapping we get

$$A_p(\underline{X}_1, H_p(\underline{X}_2, Z)) + S_p(\underline{X}_1, S_p(\underline{X}_2, Z)) + DS_p(\underline{X}_1; \underline{X}_2, Z) = 0. \qquad (27)$$

Now we shall prove that (24) and (26) are equivalent. Differentiating identity (22) with respect to $X_2 \in F$ we obtain

$$\langle Dn_p(X_2, Z), D_{i_p}(X_1)\rangle + \langle n_p(Z), D^2 i_p(X_2; X_1)\rangle = 0.$$

Substituting by (2), (3) in the above equality and using (21) we get

$$g_p(H_p(X_2, Z), X_1) + \langle n_p(Z), n_p(A_p(X_2, X_1))\rangle = 0. \qquad (28)$$

Differentiating equivalent (28) with respect to $X_3 \in F$ and using (21) we have that

$$Dg_p(X_3' H_p(X_2, Z), X_1) + g_p(DH_p(X_3; X_2, Z), X_1)$$
$$+\langle n_p(S_p(X_3, Z), n_p(A_p(X_2, X_1)) + n_p(Z), n_p(S_p(X_3, A_p(X_2, X_1)))$$
$$+n_p(DA_p(X_3, (X_3; X_2, X_1))\rangle = 0.$$

Since the fundamental metric tensor g is covariant constant, that is $\nabla g = 0$, then the above equation becomes

$$g_p(H_p(X_2, Z), \gamma_p(X_1; X_3)) + g_p(\gamma_p(H_p(X_2, Z), X_3), X_1$$
$$+g_p(DH_p(X_3, X_2, Z), X_1) + \langle n_p(S_p((X_3, Z), n_p(X_2, X_1))\rangle$$
$$+\langle n_p(Z), n_p(S_p(X_3, A_p(X_2, X_1)) + DA_p(X_3; X_2, X_1))\rangle = 0. \qquad (29)$$

Applying the alternation convention and using (26), Equation (29) takes the form

$$-g_p(H_p(\underline{X}_3, S_p(\underline{X}_2, Z)), X_1) + g_p(H_p(\underline{X}_2, Z), \gamma_p(X_1, \underline{X}_3))$$

$$+\langle n_p(S_p(\underline{X}_3, Z), n_p(\underline{X}_2, X_1))\rangle$$
$$+\langle n_p(Z), n_p(S_p(\underline{X}_3, A_p(X_2, X_1)) + DA_p(X_3; \underline{X}_2, X_1))\rangle = 0. \quad (30)$$

Replacing Z by $S_p(X_3, Z),$, X_1 by $\gamma_p(X_1, X_3)$ in (28) and substituting in (30) we have

$$\langle n_p(Z), n_p(A_p(\underline{X}_2, \gamma_p(X_1, \underline{X}_3)) + S_p(\underline{X}_3(A_p(\underline{X}_2, X_1)) + DA_p(\underline{X}_3; \underline{X}_2, X_1))\rangle = 0. \quad (31)$$

Since $n_p(Z)$ is not zero for all $Z \in W$ and the mapping n_p is injective, then from (31) we have that

$$A_p(\underline{X}_2, \gamma_p(X_1, \underline{X}_3)) + S_p(\underline{X}_3, A_p(\underline{X}_2, X_1)) + DA_p(\underline{X}_3; \underline{X}_2, X_1)\rangle = 0 \quad (32)$$

which is the equality (24) if we replace X_1 by X_2 and X_2 by X_1, i.e. the equality (24) follows from the equality (26).
On the other hand, using (24), (28) instead of (26) makes equation (29) in the form

$$g_p(\gamma_p(H_p(\underline{X}_2, Z), \underline{X}_3) + DH_p(\underline{X}_3; \underline{X}_2, Z) - H_p(\underline{X}_2; S_p(\underline{X}_3, Z)), X_1) = 0. \quad (33)$$

Since X_1 is an arbitrary vector in F and g_p is non-singular mapping, then we have that

$$\gamma_p(H_p(\underline{X}_2, Z), \underline{X}_3) + DH_p(\underline{X}_1; \underline{X}_2, Z) - H_p(\underline{X}_2; S_p(\underline{X}_3, Z)), X_1)) = 0, \quad (34)$$

which is the equality (26) if we replace X_3 by X_1. Hence the equality (26) follows from the equality (24), i.e. (24) and (26) are equivalent. Therefore Equations (23), (24) and (27) represent Gauss and Codazzi equations for a submanifold N of a Hilbert manifold M.
Let N be a hypersurface in M. Then dim $W = 1$ and therefore for every $Z \in W$ we have $Z = \lambda \cdot e$, where $\lambda \in R \cdot e \in W$ is a non-zero fixed vector in W. In this case we write

$$n_p(e) = \xi_p \in (T_x N)^{\perp}. \quad (35)$$

Moreover, without loss of generality we can choose $e \in W$ such that ξ_p is a unit normal vector. Similarly denote $H_p^*(X)$ and $A_p(X_1, X_2) = \alpha_p(X_1, X_2) \cdot e$. Hence Equation (3) has the form:

$$D\xi_p(X) = D_{i_p}(H_p^*(X)), \quad (36)$$

i.e.

$$S_p(X, Z) = 0 \quad (37)$$

and (28) will be

$$g_p(H_p^*(X_2, X_1)) = -\alpha_p(X_1, X_2). \tag{38}$$

From (35), (37), (38) the condition (37) has the form

$$\alpha_p(\underline{X}_2, H_p^*(\underline{X}_1)) \cdot e = g_p(H_p^*(\underline{X}_2), H_p^*(\underline{X}_1)) \cdot e = 0 \tag{39}$$

i.e. the condition (37) is satisfied identically. Hence Equations (23), (24) are Gauss and Codazzi equations for a hypersurface N in a Hilbert manifold M. As an application of this consideration we give the following theorem for local isometric Hilbert manifolds of constant sectional curvature.

Theorem 5.1 : A hypersurface N of constant sectional curvature β in a Hilbert inanifold M can be bent into a hypersurface if $\beta = 0$ and into a hypersurface if $\beta > 0$.

Proof : Suppose that the metric on the hypersurface N is given by the quadratic form g. Since N is a space of constant sectional curvature, then its curvature tensor has the form [las1]:

$$r_x(X_3; X_1, X_2) = \beta_x(g_x, X_2)X_1 - g_x(X_3, X_1)X_2). \tag{40}$$

For all $X_1; X_2, X_3 \in T_xN, x \in N$ and where β_x is a real function of points of N. Then from (40), it is clear that $\propto = -\sqrt{|\beta|}g$ satisfies Gauss and Codazzi equations (23), (24) such that $H^*(X) = \sqrt{|\beta|} \cdot X$.

Hence from " Bonnet theorem " [7] in the infinite dimensional case, there exists a unique (up to a transformation, which preserves the scalar product $\langle \cdot, \cdot \rangle$ in M) hypersurface S^1 which is locally isometric to N. Therefore it is easy to show that " Bonnet theorem" carries a local character when $\beta = 0$, S^1 will represent a hyperplane in M and when $\beta > 0$, S^1 will be a hypersphere in M. This is evident from the equation $\langle x = x_0, x - x_0 \rangle = \frac{1}{k}$. Since, according to equality (38), a manifold of constant sectional curvature has the same constant curvature, then this result can be restated as follows:

Theorem 5.2 : The necessary and sufficient condition for a Hilbert manifold to be locally isomorphic to a manifold of constant sectional curvature is that it has the same constant curvature.

Remark : If S^1 is a hypersphere in M, then it is shown in [las1] that:

$$\propto (X_1, X_2) = -\sqrt{|\beta|}g(X_1, X_2), \quad H^*(X) = \sqrt{|\beta|} \cdot X.$$

6. Locally Equiaffine Hypersurfaces in Hilbert Manifold

Definition 6.1 [2] : The tow hypersurfaces N_1 and N_2 of a Hilbert manifold M are called locally equiaffine if given any points $x \in N_1$ and $y \in N_2$ we can choose charts C for N_1 at x and D for N_2 at y having the same image such that the induced connections γ_1 and γ_2 with respect to the charts C, D respectively.

Theorem 6.1 : The locally equiaffine hypersurfaces of a Hilbert manifold M of constant sectional curvature are the only hyperspheres.

Proof : Let g^1 be the metric operator of a hypersphere S^1 of curvature β^1, then $g^2 = \mu g^1$ (where $\mu > 0$) can be taken as the metric operator for a hypersphere $S^2 = (S^2, g^2)$ which has curvature $\frac{\beta^1}{\mu}$. Since M is a Riemannian manifold, then we have [4]:

$$g^2(\gamma^2(X_1, X_2), X_3) = \frac{1}{2}(Dg^2(X_1; X_2, X_3) + Dg^2(X_2; X_1, X_3) - Dg^2(X_3; X_2, X_1).$$

Substituting $g^2 = \mu g^1$ into the above equation we obtain

$$\mu g^1(\gamma^2(X_1, X_2), X_3) = \mu g^1(X_1, X_2), X_3).$$

Since $\mu > 0$, g^1 is non-singular and X_3 is arbitrary vector, then

$$\gamma^2(X_1, X_2) = \gamma^1(X_1, X_2),$$

therefore $\gamma^1 = \gamma^2$, i.e. S^1 and S^2 are locally equiaffine. Conversely, suppose that S^1 and S^2 are locally equiaffine hypersurfaces in M, g^1 and g^2 are the metric operators on S^1 and S^2 resp., then we can choose two charts C on S^1 and D on S^2 such that $\gamma^1 = \gamma^2$, which leads to $\gamma^1 = \gamma^2$, therefore from Gauss equation (4) we have

$$\propto_p^1 (\underline{X}_1, X_2) H_p^{*1}(\underline{X}_3) = \propto_p^2 (\underline{X}_1, X_2) H_p^{*2}(\underline{X}_3). \tag{41}$$

Let X_3 be an arbitrary vector in M such that $\propto_p^1 (X_2, X_0) = \propto_p^2 (X_3, X_0) = 0$, then (41) has the form:

$$\propto_p^1 (X_2, X_0) H_p^{*1}(X_3) = \propto_p^2 (X_2, X_0) H_p^{*2}(X_3). \tag{42}$$

If $\propto^1$ is the operator of the second fundamental form of a hypersphere S^1 then from our Remark we get

$$\begin{aligned} \propto_p^1 (X_2, X_0) &= -\sqrt{|\beta^1|} g_p^1(X_2, X_0), H_p^{*1}(X_3) = \sqrt{|\beta^1|} \cdot X_3, \\ &\quad -\sqrt{|\beta^1|} g_p^1(X_2, X_0) \cdot X_3 = \propto_p^2 (X_2, X_0) H_p^{*2}(X_3). \end{aligned}$$

Multiplication both sides of the above equation by $X \in S^2$ with respect to the metric g^2 and using (39) yields

$$\begin{aligned} -\sqrt{|\beta^1|} g_p^1(X_2, X_0) g_p^2(X_3, X) &= \propto_p^2 (X_2, X_0) g_p^2(H_p^{*2}(X_3), X \\ &= - \propto^2 (X_2, X_0) \propto_p^2 (X, X_3). \end{aligned}$$

Suppose that at $g_p^1(X_0, X_0) = 1, \propto_p^2 (X_0, X_0) = \lambda, \lambda \neq 0$. Hence $\sqrt{|\beta^1|} g_p^2(X_3, X) = \lambda \alpha_p^2(X, X_3)$, from which we get

$$\propto_p^2 (X, X_3) = \frac{\sqrt{\beta^1|}}{\lambda} g_p^2(X, X_3).$$

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