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THIRD HANKEL DETERMINANT FOR NEW SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract

In this paper we have obtained an upper bound of third Hankel determinant for the analytic functions belonging to the classes $M(\alpha)$ and $N(\lambda)$ and all results are sharp.

1. Introduction

Let A denote the class of normalized, analytic and univalent function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{where} \quad z \in U = \{z : |z| < 1\}. \tag{1.1}$$

The q^{th} Hankel determinant for $q \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas as

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$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}$$

In this paper we consider the case $q = 1, n = 3$ that is third Hankel determinant,

$$H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

By using triangle inequality,

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.2}$$

Definition 01 : Let $f(z)$ is given by (1.1). Then $f(z) \in M(\alpha)$ if and only if

$$Re\left\{\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)}\right\} > 0, \quad 0 \leq \alpha \leq 1, \quad z \in U. \tag{1.3}$$

We see that $M(0) = S^*$, class of starlike functions.

$M(1) = C$, class of convex functions.

Definition 02 : Let $f(z)$ is given by (1.1). Then $f(z) \in N(\lambda)$ if and only if

$$Re\left\{\frac{\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)}{\lambda z^2 f''(z) + zf'(z)}\right\} > 0, \quad 0 \leq \lambda \leq 1, \quad z \in U. \tag{1.4}$$

We note that $N(0) = C$, class of convex functions.

These classes studied by T. Tulsiram, K. Suchithra and R. Sattanathan [8] for second Hankel determinant. In this paper, we consider third Hankel determinant and obtain an upper bound to the functional $H_3(1)$ for the functions in the classes $M(\alpha)$ and $N(\lambda)$.

2 Preliminary Results

Let A be the family of all functions $P(z)$ analytic in U for which $Re(P(z)) > 0$ and $P(z) = 1 + p_1z|p_2z^2 + \cdots$ for $z \in U$.

Lemma 2.1 [5] : If $P \in A$ then, $|p_k| \leq 2 \quad (k = 1, 2, 3, \cdots)$.

Lemma 2.2 ([6],[7]) : If $P \in A$ then,

$$\begin{aligned} 2p_1 &= p_1^2 + (4 - p_1^2) \\ 4p_3 &= p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z \end{aligned}$$

for some x and z satisfying $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

Lemma 2.3 ([2],[3]) : If $P \in A$ then,

$$\left| p_1^2 + p_2 - \sigma \frac{p_1^2}{2} \right| = \begin{cases} 2(1-\sigma) & if \quad \sigma \leq 0 \\ 2 & if \quad 0 \leq \sigma \leq 2 \\ 2(\sigma - 1) & if \quad \sigma \geq 2 \end{cases}$$

Lemma 2.4 ([2],[3]) : If $P \in A$ then

$$\left| p_1^2 + p_2 - \mu \frac{p_1^2}{2} \right| = \begin{cases} 2(1-\mu) & if \quad \mu \leq 0 \\ 2 & if \quad 0 \leq \mu \leq 2 \\ 2(\mu - 1) & if \quad \mu \geq 2 \end{cases}$$

Lemma 2.5 [8] : If $f \in M(\alpha)$ then $|a_2 a_4 - a_3^2| \leq \frac{1}{(1+\alpha)(1+3\alpha)}$.

Lemma 2.6 [8] : If $f \in N(\lambda()$ then $|a_2 a_4 - a_3^2| \leq \frac{1}{8(1+\lambda)(1+3\lambda)}$.

3. Main Results

Theorem 3.1 : If $f \in M(\alpha)$ then

$$\begin{aligned} |a_2| &\leq \frac{p_1}{(1+\alpha)} \qquad\qquad |a_3| \leq \frac{p_1^2 + p_2}{2(1+2\alpha)} \\ |a_4| &\leq \frac{p_1^3}{6(1+3\alpha)} + \frac{p_1 p_2}{2(1+3\alpha)} + \frac{p_3}{3(1+3\alpha)} \\ |a_5| &\leq \frac{p_1^4}{24(1+4\alpha)} + \frac{p_1 p_3}{3(1+4\alpha)} + \frac{p_2^2}{8(1+4\alpha)} + \frac{p_1^2 p_2}{4(1+4\alpha)} + \frac{p_4}{4(1+4\alpha)}. \end{aligned}$$

Proof : Since $f(z) \in M(\alpha)$ then there exit $P \in A$ such that,

$$\frac{zf'(z) + az^2 f''(z)}{(1-\alpha)f(z) + azf'(z)} = P(z)$$

$$zf'(z) + \alpha z^2 f''(z) = P(z)[(1-\alpha)f(z) + \alpha z f'(z)] \tag{3.1}$$

Equating the coefficients in (3.1) yields

$$|a_2| \leq \frac{p_1}{(1+\alpha)} \tag{3.2}$$

$$|a_3| \leq \frac{p_1^2 + p_2}{2(1+2\alpha)} \tag{3.3}$$

$$|a_4| \leq \frac{p_1^3}{6(1+3\alpha)} + \frac{p_1 p_2}{2(1+3\alpha)} + \frac{p_3}{3(1+3\alpha)} \tag{3.4}$$

$$|a_5| \leq \frac{p_1^4}{24(1+4\alpha)} + \frac{p_1 p_3}{3(1+4\alpha)} + \frac{p_2^2}{8(1+4\alpha)} + \frac{p_1^2 p_2}{4(1+4\alpha)} + \frac{p_4}{4(1+4\alpha)} \tag{3.5}$$

Theorem 3.2 : If $f \in M(\alpha)$ then

$$|a_2 a_3 - a_4| \leq \frac{12\alpha^2 + 9\alpha + 8}{6(1+\alpha)(1+2\alpha)(1+3\alpha)}. \tag{3.6}$$

Proof : From equations (3.2), (3.3) and (3.4), we obtain

$$|a_2 a_3 - a_4| = \left| \begin{array}{c} \frac{p_1^3(1+3\alpha-\alpha^2)}{3(1+\alpha)(1+2\alpha)(1+3\alpha)} - \frac{p_1\alpha^2(p_1^2+(4-p_1^2)x)}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} \\ \\ -\frac{p_1^3+2p_1(4-p_1^2)x-p_1(4-p_1^2)x^2+2(4-p_1^2)(1-|x|^2)z}{12(1+3\alpha)} \end{array} \right|. \tag{3.7}$$

Since $|p_1| \leq 2$ from Lemma 2.1, we get $p_1 = p$ and assume without restriction that $p \in [0,2]$. Thus applying triangle inequality on (3.7) with $|z| \leq 1$ and $\rho = |x|$, we obtain

$$\begin{array}{rl} & \frac{p^3(1+3\alpha-\alpha^2)}{3(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{p\alpha^2(p^2+(4-p^2)\rho)}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} \\ |a_2 a_3 - a_4| = & +\frac{[p^3+2p(4-p^2)\rho+(4-p^2)(p-2)\rho^2+2(4-p^2)]}{12(1+3\alpha)} \quad . \\ & = F(\rho) \end{array} \tag{3.8}$$

Furthermore,

$$f'(\rho) = \frac{p\alpha^2(4-p^2)}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{1}{12}\frac{[2p(4-p^2) + 2\rho(4-p^2)(p-2)]}{(1+3\alpha)} \tag{3.9}$$

It can be easily shown that, $F'(\rho) > 0$ and thus, is an increasing function implying $\max_{\rho\leq 1}^{f(\rho)=F(1)}$.

Now let,

$$\begin{aligned} G(P) &= F(1) \\ &= \frac{p^3(1+3\alpha-\alpha^2)}{3(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{4p\alpha^2}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{p^3+3p(4-p^2)}{12(1+3\alpha)}. \end{aligned}$$

Trivially one can show $G(p)$ has maximum attained at $p = 1$. The upper bound for $|a_2a_3 - a_4|$ is attained for $\rho = 1$ and $p = 1$. That is

$$|a_2a_3 - a_4| \leq \frac{12a^2 + 9\alpha + 8}{6(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

Theorem 3.3 : If $f \in M(\alpha)$ then, $|a_3 - a_2^2| \leq \frac{1}{(1+2\alpha)}$.

Proof : If $f \in M(\alpha)$ then from equation (3.2) and (3.3) we obtain,

$$\begin{aligned} |a_3 - a_2^2| &= \left| \frac{p_1^2 + p_2}{2(1+2\alpha)} - \frac{p_1^2}{(1+\alpha)^2} \right| \\ &= \frac{1}{2(1+2\alpha)} \left| p_1^2 + p_2 - \frac{4(1+2\alpha)}{(1+\alpha)^2} \frac{p_1^2}{2} \right|. \end{aligned}$$

Using Lemma 2.3 with $0 \leq \sigma = \frac{4(1+2\alpha)}{(1+\alpha)^2} \leq 2$.

We have, $|a_3 - a_2^2| \leq \frac{1}{(1+2\alpha)}$.

Theorem 3.4 : If $f \in M(\alpha)$ then, $|H_3(1)| \leq \frac{231\alpha^3+363\alpha^2+250\alpha+40}{3(1+\alpha)(1+2\alpha)(1+3\alpha)^2(1+4\alpha)}$.

Proof : Using Lemma 2.5 and Theorem (3.1), (3.2) and (3.3) in inequality (1.2), the above result can be easily obtained.

For $\alpha = 0$, Theorem 3.4 gives following result:

Corollary 3.4.1 : If $f \in M$ then,

$$|H_3(1)| \leq \frac{40}{3}.$$

Corollary 3.4.2 : If $f \in M$ then,

$$|H_3(1)| \leq \frac{221}{360}.$$

Theorem 3.5 : If $f \in N(\lambda)$ then,

$$\begin{aligned} |a_2| &= \frac{p_1}{2(1+2\lambda)} \qquad |a_3| = \frac{p_1^2 + p_2}{6(1+2\lambda)} \\ |a_4| &= \frac{p_1^3}{24(1+3\lambda)} + \frac{p_1p_2}{8(1+3\lambda)} + \frac{p_3}{12(1+3\lambda)} \\ |a_5| &= \frac{1}{20(1+4\lambda)} \left(\frac{p_1^4}{6} + p_1^2p_2 + \frac{4p_1p_3}{3} + \frac{p_2^2}{2} + p_4 \right). \end{aligned}$$

Proof : Let $f \in N(\lambda)$ then there exist $P \in A$ such that,

$$[\lambda z^3 f'''(z) + (1+2\lambda)z^2 f''(z) + zf'(z)] = [\lambda z^2 f''(z) + zf'(z)]P(z). \tag{3.10}$$

Equating the coefficients in (3.10) yields,

$$|a_2| = \frac{p_1}{2(1+2\lambda)} \tag{3.11}$$

$$|a_3| = \frac{p_1^2 + p_2}{6(1+2\lambda)} \tag{3.12}$$

$$|a_4| = \frac{p_1^3}{24(1+3\lambda)} + \frac{p_1 p_2}{8(1+3\lambda)} + \frac{p_3}{12(1+3\lambda)} \tag{3.13}$$

$$|a_5| = \frac{1}{20(1+4\lambda)} \left(\frac{p_1^4}{6} + p_1^2 p_2 + \frac{4p_1p_3}{3} + \frac{p_2^2}{2} + p_4 \right). \tag{3.14}$$

Theorem 3.6 : If$f \in N(\lambda)$ then,

$$|a_2a_3 - a_4| \le \frac{62\lambda^2 + 75\lambda + 25}{24(1+\lambda)(1+2\lambda)(1+3\lambda)}. \tag{3.15}$$

Proof : From equations (3.11), (3.12) and (3.13), we obtain

$$|a_2a_3 - a_4| = \left| \begin{array}{l} \frac{p_1^2(1+3\lambda-2\lambda^2)}{24(1+\lambda)(1+2\lambda)(1+3\lambda)} - \frac{p_1(1+3\lambda+6\lambda^2)(p_1^2+(4-p_1^2)x)}{24(1+\lambda)(1+2\lambda)(1+3\lambda)} \\ \\ -\frac{p_1^3+2p_1(4-p_1^2)x-p_1(4-p_1^2)x^2+2(4-p_1^2)(1-|x|^2)z}{12(1+3\lambda)} \end{array} \right| \tag{3.16}$$

Since $|p_1| \le 2$ from Lemma (2.1), we get $p_1 = p$ and assume without restriction that $p \in [0,2]$. Thus applying triangle inequality on (3.16) with $|z| \le 1$ and $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &= \begin{array}{l} \frac{p^3(1+3\lambda-2\lambda^2)}{24(1+\lambda)(1+2\lambda)(1+3\lambda)} + \frac{p(1+3\lambda+6\lambda^2)(p_1^2+(4-p_1^2)\rho)}{24(1+\lambda)(1+2\lambda)(1+3\lambda)} \\ \\ +\frac{[p^3+2p(4-p^2)\rho+(4-p^2)(p-2)\rho^2+2(4-p^2)]}{12(1+3\lambda)} \end{array} \\ &= F(\rho). \end{aligned} \tag{3.17}$$

Furthermore,

$$F'(\rho) = \frac{p(1+3\lambda+6\lambda^2)}{6(1+\lambda)(1+2\lambda)(1+3\lambda)} + \frac{[p(4-p^2)+\rho(4-p^2)(p-2)]}{6(1+3\lambda)}.$$

It can be easily shown that, $F'(\rho) > 0$ and thus, is an increasing function implying $\max_{\rho\le 1}^{F(\rho)=F(1)}$.

Now let, $G(P) = F(1)$

$$= \frac{p^3(1+3\lambda-2\lambda^2)}{24(1+\lambda)(1+2\lambda)(1+3\lambda)} + \frac{4p(1+3\lambda+6\lambda^2)}{24(1+\lambda)(1+2\lambda)(1+3\lambda)} + \frac{p^3+3p(4-p^2)}{12(1+3\lambda)}. \tag{3.18}$$

Trivially one can show $G(p)$ has maximum attained at $p = 1$. The upper bound for $|a_2a_3 - a_4|$ is attained for $\rho = 1$ and $p = 1$. That is

$$|a_2a_3 - a_4| \leq \frac{62\lambda^2 + 75\lambda + 25}{24(1+\lambda)(1+2\lambda)(1+3\lambda)}.$$

Theorem 3.7 : If $f \in N(\lambda)$ then, $|a_3 - a_2^2| \leq \frac{1}{4(1+3\lambda)}$.

Proof : If $f \in N(\lambda)$ then from equation (3.11) and (3.12) we obtain,

$$\begin{aligned} |a_3 - a_2^2| &= \left| \frac{p_1^2 + p_2}{8(1+3\lambda)} - \frac{p_1^2}{4(1+\lambda)^2} \right| \\ &= \frac{1}{8(1+3\lambda)} \left| p_1^2 + p_2 - \frac{4(1+3\lambda)}{(1+\lambda)^2} \frac{p_1^2}{2} \right|. \end{aligned}$$

Using Lemma 2.4 with $0 \leq \sigma = \frac{4(1+3\lambda)}{(1+\lambda)^2} \leq 2$.

We have, $|a_3 - a_2^2| \leq \frac{1}{4(1+3\lambda)}$.

Theorem 3.8 : If $f \in N(\lambda)$ then,

$$|H_3(1) \leq \frac{142\lambda^3 + 232\lambda^2 + 116\lambda + 17}{12(1+\lambda)(1+2\lambda)(1+3\lambda)^2(1+4\lambda)}.$$

Proof : Using Lemma 2.6 and Theorem (3.5), (3.6) and (3.7) in inequality (1.2), the above result can be easily obtained.

For $\lambda = 0$, Theorem (3.8) gives following result:

Corollary 3.8.1 : If $f \in N$ then,

$$|H_3(1)| \leq \frac{17}{12}.$$

Corollary 3.8.2 : If $f \in N$ then,

$$|H_3(1)| \leq \frac{169}{1920}.$$

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