International J. of Pure & Engg. Mathematics (IJPEM)
ISSN 2348-3881, Vol. 5 No. III (December, 2017), pp. 9-37

LOWER AND UPPER LIMITS OF MEAN OF STATISTICAL DISTRIBUTION

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Abstract

Belief function and plausibility function give one of the pair of lower and upper limits of probability function respectively. We have another basic belief assignment based on probability mass function. In this paper, belief function and plausibility function obtained from this basic belief assignment, are used to obtain lower and upper limits of distribution function and mean of probability distributions. Also we tried to obtain these limits in terms of probabilities given by probability mass function of probability distribution by using series results [6].

1. Introduction

In [3,4], a special case of upper and lower probabilities has been introduced by Dempster. Existence of probability function is assumed, which is one to many mapping m from space X to frame of discernment Θ. The lower probability of A in X is equal to the probability of the largest subset of Θ such that its image under m is included in A. The upper probability of A in space X such that the image under m of all elements have a

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Key Words : *Belief function, Plausibility function, Probability, Probability mass function.*
AMS Subject Classification : 03B42.

non-empty intersection with A. In [11], belief functions on a system of sets of an infinite or finite universe are represented by a probability measure or probability charge. In Kyburg's article [8], let set Π of all those probability distributions compatible with the available information

$$\forall \quad A \subseteq \Theta, P^*(A) = inf_{p\in\Pi} p(A) \quad P_*(A) = sup_{p\in\Pi} p(A) \tag{1}$$

with Π is a convex set of probability distributions. For confidence bands, $\underline{F}(x) \leq F(x) \leq \overline{F}(x)$, where $F(x)$ is not precisely known and we can specify $\underline{F}(x)$ and $\overline{F}(x)$ from $\mathcal{R}$ to $[0, 1]$. Then the distribution band is $\Gamma(\underline{F}, \overline{F}) = (F|\forall \quad x \in \mathcal{R}, \underline{F}(x) \leq F(x) \leq \overline{F}(x))$ [7]. If $\underline{F}(x)$ and $\overline{F}(x)$ are step functions then distribution band becomes probability box [5]. In [2], imprecise belief structures are set of belief structures whose masses on focal elements A, interval-valued constraints $M = \{m : a_i \leq m(A_i) \leq b_i\}$. The intervals $[a_i, b_i]$ specifying an imprecise belief structures are not unique if $m(A_i) \leq min.\{b_i, 1-\sum_{j\neq i} a_j\}$. Upper and lower bounds to m determine interval ranges for belief and plausibility functions. Yager [12] considers same situations in which the masses of focal elements lie in some known interval, allowing us to model realistically situation in which the basic probability assignments can not be precisely identified.

In this paper, we calculate distribution function and mean of any given probability distribution, if possible. By Shafer‘s basic belief assignment [10], probability of set, belief of set and plausibility of set are equal. But in our defined basic belief assignment, $Bel(A) \leq p(A) \leq Pl(A), \quad \forall A \subseteq \Theta$ which is mentioned in Dempster‘s articles [3, 4]. Hence we calculate lower and upper limits of distribution function and mean of any given probability distribution (sections 3-7). Here, we obtain lower and upper limits of distribution function and mean by using steps:

1 Calculate distribution function and mean with help of given probability distribution.

2 Calculate lower and upper limits of already calculated distribution function and mean by using belief functions and plausibility functions as lower and upper limits of probability. Also we calculate distribution function and mean based on probability of set.

3 Dividing by distribution function and mean based on probability of set to distribution function and mean by using belief and plausibility functions and multiply by distribution function and mean of distribution (by considering mutually disjoint and exhaustive subsets of Θ). Thus, we get lower and upper limits of distribution function and mean (by considering mutually disjoint and exhaustive subsets of Θ).

2. Preliminaries

In this section,we provide necessary preliminaries about discrete belief function theory [10], interval arithmetic [9] and discrete probability distribution theory [1].

2.1 Discrete Belief Function Theory [10]

Frame of Discernment :

Dictionary meaning of Frame of Discernment is frame of good judgement insight. The word discern means recognize or find out or hear with difficulty. If frame of discernment Θ is

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$$

then every element of Θ is a proposition. The propositions of interest are in one -to -one correspondence with the subsets of Θ. The set of all propositions of interest corresponds to the set of all subsets of Θ, denoted by 2^Θ. A function $m : 2^\Theta \to [0, 1]$ is called **basic probability assignment** whenever

1. $m(\emptyset) = 0$.
2. $\sum_{A\subset\Theta} m(A) = 1$.

The quantity $m(A)$ is called A's **basic probability number** and it is a measure of the belief committed exactly to A. The *total belief* committed to A is sum of $m(B)$, for all proper subsets B of A.

$$Bel(A) = \sum_{B\subset A} m(B). \tag{2}$$

If Θ is a frame of discernment, then a function $Bel : 2^\Theta \to [0, 1]$ is called **belief function** over Θ if it satisfies above condition (2). A function $Bel : 2^\Theta \to [0, 1]$ is *belief function* if and only if it satisfies following conditions

1. $Bel(\emptyset) = 0$.

2. $Bel(\Theta) = 1$.

3. For every positive integer n and every collection $A_1, A_2, \ldots, A_n$ of subsets of Θ

$$Bel(A_1 \cup A_2 \cup \ldots \cup A_n) \geq \sum_{I \subset \{1,2,\cdots,n\}} (-1)^{|I|+1} Bel(\bigcap_{i \in I} A_i). \quad (3)$$

Degree of doubt :

$$Dou(A) = Bel(\bar{A}) \text{ or } Bel(A) = Dou(\bar{A}). \quad (4)$$

The quantity $pl(A) = 1 - Dou(A) = \sum_{A \cap B \neq \emptyset} m(B)$ which expresses the extent to which one finds A credible or plausible. We have relation between belief function, probability mass (or density) function and plausibility function [3, 4] as:

$$Bel(A) \leq p(A) \leq Pl(A), \quad \forall A \subset \Theta. \quad (5)$$

In Moore's book [9], operations on intervals viz. addition, subtraction, multiplication, division and functions on intervals are explained in detail. In division of intervals, if union of intervals is $(-\infty, \infty)$ then it is better to perform calculations and draw conclusion on separate intervals whose union is $(-\infty, \infty)$. From Bansi Lal and Sanjay Arora's book [1], we have referred preliminaries about distribution function and mean of discrete probability distributions.

3. Lower and Upper Limits of Distribution Function

If $|\Theta| = n$ then every element in frame of discernment Θ is repeated exactly 2^{n-1} number of times and sum of probabilities of all subsets of Θ is 2^{n-1}. Now, let $A = \{\{a_1\}, \{a_2\}, \ldots, \{a_n\}\} \subseteq \Theta$. In discrete space, since singletons are disjoint, the intersection of any number of singleton subsets of Θ is always emptyset. Therefore we get

$$m(A) = \frac{p(A)}{2^{n-1}}, \forall A \subseteq \Theta. \quad (6)$$

The quantity $m(A) = \dfrac{P(A)}{2^{n-1}}$ is a basic probability assignment.

3.1 Indexing of Subsets of Θ

In order to apply stastistical concepts for our defined basic belief assignment, we apply indexing of subsets of Θ as follows:

Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ hence $|\Theta| = n$. Number of subsets of Θ are 2^n. We define indicator function as :

$$\text{For any subset } A \text{ of } \Theta, I_A(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \notin A \\ 1 & \text{if } \theta_i \in A. \end{cases} \tag{7}$$

If $A = \{\theta_j, \theta_k, \theta_l, \theta_m, \theta_p, \theta_q\}$ then indexing number of A in Θ is

$$v = \sum_{i=1}^{n} I_A(\theta_i) 2^{i-1} = 2^{j-1} + 2^{k-1} + 2^{l-1} + 2^{m-1} + 2^{p-1} + 2^{q-1}$$

Notes :-

1 $0 \le v \le 2^n - 1$.

2 $v = 0$ corresponds to $\emptyset$.

3 $v = 2^n - 1$ corresponds to Θ.

4 Any value in between 0 and $2^n - 1$ corresponds to proper subset of Θ.

5 Indexing of subsets of Θ is helpful in obtaining statistical quantities as it does not affect results of statistics and mathematics.

With this indexing of set, we will obtain some statistical quantities from Bansi Lal and Arora Sanjay‘s book [1] as :

1 **Distribution Function:** $P(v) = P[V \le v] = \sum_{V=0}^{n} p(V)$.

2 **Expectation of V =Mean:** $E(V) = \sum_{V=0}^{n} V p(V)$.

With this indexing, we have following observations as:

1 As number of elements or subsets of Θ increases for indexing i.e V increases, value of statistical quantities also increases.

2 If indexing of elements or subsets of Θ is altered then value of statistical quantities is also altered accordingly. If values of V are assigned according to size of subset and subscript of variables θ_i, in ascending order then we get suitable value of statistical quantities. If values of V are assigned according to probability of subset, in descending order then we get smallest value of statistical quantities. If values of V are assigned according to probability of subset and subscript of variables θ_i, in ascending order then we get largest value of statistical quantities.

With these observations, we should choose indexing of subsets of Θ according to our interest.

Now, we calculate distribution function $P(X \leq k)$ of given probability distribution with help of probability mass function. For $k = 0, 1, 2, \ldots, n$ and by formula

$$\frac{\sum_{V=0}^{2^k-1} Bel(V)}{\sum_{V=0}^{2^k-1} P(V)} P(X \leq k) \leq P(X \leq k) \leq \frac{\sum_{V=0}^{2^k-1} Pl(V)}{\sum_{V=0}^{2^k-1} P(V)} P(X \leq k), \tag{8}$$

we get lower and upper limits of distribution function.

In another way, we can calculate lower and upper limits of probability distribution as: By indexing of subsets of Θ, we have

X	:	0	1	2	3	4
V	:	1	2	4	8	16

Now subsets of Θ required for probability distribution, have relation between X and V as:

Sr. No.	*Subset of* Θ	V
1	$\{x_0\}$	1
2	$\{x_0, x_1\}$	1+2=3
3	$\{x_0, x_1, x_2\}$	1+2+4=7
4	$\{x_0, x_1, x_2, x_3\}$	1+2+4+8=15
5	$\{x_0, x_1, x_2, x_3, x_4\}$	1+2+4+8+16=31
$\vdots$	$\vdots$	$\vdots$
k	$\{x_0, x_1, x_2, \ldots, x_k\}$	$1 + 2 + 4 + \cdots + 2^k = 2^{k+1} - 1$

Now by notation

$$p(v) = p(A_v) \quad v = 0, 1, 2, 3, \cdots, 2^n - 1$$

By indexing of sets, $F(x) = P(X \leq x) = p(\{0, 1, 2, 3, \ldots, n\}) = p(A_v)$ and only in this case, relation between x and v is $v = 2^{x+1} - 1, \quad x = 0, 1, 2, 3, \ldots, n$. By lower and upper limits of probability of sets, $Bel(A_v) \leq P(A_v) \leq Pl(A_v)$, we get $Bel(A_v) \leq F(X) \leq x) \leq Pl(A_v), \quad x = 0, 1, 2, 3, \ldots, n$ and $v = 2^{x+1} - 1$ Therefore we get lower and upper limits of distribution function of given probability distribution including the case of subset $\emptyset$.

4. Lower and Upper Limits of Statistical Quantity of Probability Distribution

Here we calculate statistical quantity of given probability distribution with help of probability mass function. Similar to distribution function, we can obtain lower and upper limits of respective statistical quantity as:

$$\frac{Stat.Quan._{Bel}}{Stat.Quan._{prob.}} Stat.Quan._{pd} \leq Stat.Quan._{pd} \leq \frac{Stat.Quan._{Pl}}{Stat.Quan._{prob.}} Stat.Quan._{pd} \tag{9}$$

where

$Stat.Quan._{Bel} =$ Statistical quantity based on belief function of set

$Stat.Quan._{prob.} =$ Statistical quantity based on probability of set

$Stat.Quan._{pd} =$ Statistical quantity based on probability mass function of probability distribution

$Stat.Quan._{Pl} =$ Statistical quantity based on plausibility function of set.

5. Calculation of Statistical Quantities based on Probability of set

Using formlae from Hall and Knight book [6], we find sums of some finite series as per our requirement. Consider

$$\begin{aligned}
\sum_{V=0}^{2^n-1} VP(V) &= 0P(0) + 1P(1) + 2P(2) + 3P(3) \\
&+ 4P(4) + 5P(5) + 6P(6) + 7P(7) \\
&+ 8P(8) + 9P(9) + 10P(10) + 11P(11) \\
&+ 12P(12) + 13P(13) + 14P(14) + 15P(15) \\
&+ \vdots \\
&+ \cdots + (2^n - 1)P(2^n - 1) \\
&= 0P(\emptyset) + 1P(\{x_1\}) + 2P(\{x_2\}) + 3(P(\{x_1\}) + P(\{x_2\})) \\
&+ 4P(\{x_3\}) + 5(P(\{x_1\}) + P(\{x_3\})) + 6(P(\{x_2\}) + P(\{x_3\})) \\
&+ 7(P(\{x_1\}) + P(\{x_2\}) + P(\{x_3\})) + 8P(\{x_4\}) + 9(P(\{x_1\} + P(\{x_4\})) \\
&+ 10(P(\{x_2\} + P(\{x_4\})) + 11(P(\{x_1\}) + P(\{x_2\}) + P(\{x_4\})) \\
&+ 12(P(\{x_3\}) + P(\{x_4\})) + 13(P(\{x_1\}) + P(\{x_3\}) + P(\{x_4\})) \\
&+ 14(P(\{x_2\} + P(\{x_3\}) + P(\{x_4\})) \\
&+ 15(P(\{x_1\}) + P(\{x_2\}) + P(\{x_3\}) + P(\{x_4\})) \\
&+ \vdots \\
&+ \cdots + (2^n - 1)(P(\{x_1\} + P(\{x_2\} + \cdots + P(\{x_n\}))
\end{aligned} \tag{10}$$

$$\begin{aligned}
\sum_{V=0}^{2^n-1} VP(V) &= (1 + 3 + 5 + 7 + \cdots + 2^n - 1)P(\{x_1\}) \\
&+ (2 + 3 + 6 + 7 + 10 + 11 + 14 + 15 + \cdots + (2^n - 2) + (2^n - 1))P(\{x_2\}) \\
&+ (4 + 5 + 6 + 7 + 12 + 13 + 14 + 15 + \cdots + (2^n - 4) + (2^n - 3) \\
&+ (2^n - 2) + (2^n - 1))P(\{x_3\}) \\
&+ \vdots \\
&+ ((2^{n-1}) + (2^{n-1} + 1) + (2^{n-1} + 2) + \cdots + (2^{n-1} + (2^{n-1} - 1)))P(\{x_n\}).
\end{aligned} \tag{11}$$

Now we will calculate coefficients of $P(\{x_j\}), j = 1, 2, \ldots, n$ as:

$$\begin{aligned}\text{Consider } & 1 + 3 + 5 + 7 + \cdots + (2^n - 1) \\ & = \left(2^{n-1}\right)^2.\end{aligned} \tag{12}$$

$$\begin{aligned}
&\text{Consider } 2 + 3 + 6 + 7 + 10 + 11 + 14 + 15 + \cdots + (2^n - 2) + (2^n - 1) \\
&= 2 + 3 + 2(4) + 2 + 3 + 2(8) + 2 + 3 + 2(12) + 2 + 3 + 2(16) + 2 + 3 \\
&+ \cdots + 8(2^{n-3} + 2^{n-4} + \cdots + 2^2 + 2^1 + 2^0) \\
&= \sum_{k=0}^{2^{n-3}+2^{n-4}+\cdots+2^2+2^1+2^0} 8k + (2+3) \\
&= (2+3) + \sum_{k=1}^{2^{n-3}+2^{n-4}+\cdots+2^2+2^1+2^0} 8k + (2+3) \\
&= (2+3) + (2+3)(2^{n-3} + 2^{n-4} + \cdots + 2^2 + 2^1 + 2^0) \\
&+ 8(2^{n-3} + 2^{n-4} + \cdots + 2^2 + 2^1 + 2^0)(2^{n-3} + 2^{n-4} + \cdots + 2^2 + 2^1 + 2^0 + 1)/2 \\
&= 5 + 5(2^{n-2} - 1)(2^n + 5) \\
&= 2^0(5 + (2^{n-2} - 1)(2^n + 5)).
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\text{Consider } 4 + 5 + 6 + 7 + 12 + 13 + 14 + 15 + \cdots + (2^n - 4) + (2^n - 3) + (2^n - 2) + (2^n - 1) \\
&= (4 + 5 + 6 + 7) + 4(8) + (4 + 5 + 6 + 7) + 4(16) + (4 + 5 + 6 + 7) \\
&+ 4(24) + (4 + 5 + 6 + 7) + \cdots + 32(2^{n-4} + 2^{n-5} + \cdots + 2^2 + 2^1 + 2^0) \\
&= \sum_{k=0}^{2^{n-4}+2^{n-5}+\cdots+2^2+2^1+2^0} (4)(8)k + (4 + 5 + 6 + 7) \\
&= (4 + 5 + 6 + 7) + \sum_{k=1}^{2^{n-4}+2^{n-5}+\cdots+2^2+2^1+2^0} (4)(8)k + (4 + 5 + 6 + 7) \\
&= (4 + 5 + 6 + 7) + 22(2^{n-4} + 2^{n-5} + \cdots + 2^2 + 2^1 + 2^0) \\
&+ 32(2^{n-4} + 2^{n-5} + \cdots + 2^2 + 2^1 + 2^0)(2^{n-4} + 2^{n-5} + \cdots + 2^2 + 2^1 + 2^0 + 1)/2 \\
&= 22 + (2^{n-3} - 1)(22 + 2(2^n)) \\
&= 2^1(11 + (2^{n-3} - 1)(11 + 2^n)).
\end{aligned} \tag{14}$$

Consider
$$
\begin{aligned}
&8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 24 + 25 + 26 + 27 + 28 + 29 + 30 + 31\\
&+ \cdots + (2^n - 8) + (2^n - 7) + (2^n - 6) + (2^n - 5) + (2^n - 4)\\
&+ (2^n - 3) + (2^n - 2) + (2^n - 1)\\
&= (8 + 9 + 10 + 11 + 12 + 13 + 14 + 15)\\
&+ 16(8) + (8 + 9 + 10 + 11 + 12 + 13 + 14 + 15)\\
&+ 32(8) + (8 + 9 + 10 + 11 + 12 + 13 + 14 + 15)\\
&+ 48(8) + (8 + 9 + 10 + 11 + 12 + 13 + 14 + 15)\\
&+ \cdots + 96 + (8 + 9 + 10 + 11 + 12 + 13 + 14 + 15)\\
&= \sum_{k=0}^{2^{n-5}+2^{n-6}+\cdots+2^2+2^1+2^0} (16)(8)k + (8 + 9 + 10 + 11 + 12 + 13 + 14 + 15)\\
&= 4(23) + \sum_{k=1}^{2^{n-5}+2^{n-6}+\cdots+2^2+2^1+2^0} (16)(8)k + (8 + 9 + 10 + 11 + 12 + 13 + 14 + 15)\\
&= 4(23) + 4(23)(2^{n-5} + 2^{n-6} + \cdots + 2^2 + 2^1 + 2^0)\\
&+ (16)(8)(2^{n-5} + 2^{n-6} + \cdots + 2^2 + 2^1 + 2^0)(2^{n-5} + 2^{n-6} + \cdots + 2^2 + 2^1 + 2^0 + 1)/2\\
&= 4\{23 + (2^{n-4} - 1)(23 + 2^n)\}\\
&= 2^2\{23 + (2^{n-4} - 1)(23 + 2^n)\}.
\end{aligned}
\tag{15}
$$

By analogy, the coefficient of $P(\{x_j\})$ is

$$
\sum_{k=0}^{K^{(j)}} (2^{j-1})(2^j)k + (2^{j-1} + (2^{j-1} + 1) + (2^{j-1} + 2) + \cdots + (2^{j-1} + (2^{j-1} - 1)
$$

where$K^{(j)} = 2^{n-(j+1)} + 2^{n-(j+1)-1} + \cdots + 2^2 + 2^1 + 2^0$ (16)

$$= (2^j)(2^{j-1})k + 2^{j-2}(2^{j-2} + 2^{j-1} - 1)$$

$$= (2^{j-2})(2^{n-j} - 1)(2^{j-2} + 2^{j-1} - 1 + 2^n) + 2^{j-2}(2^{j-2} + 2^{j-1} - 1).$$

The coefficient of $P(\{x_{n-1}\})$ is

$$\sum_{k=0}^{K^{(n-1)}} (2^{n-2})(2^{n-1})k + (2^{n-2} + (2^{n-2}+1) + (2^{n-2}+2) + \cdots + (2^{n-2} + (2^{n-2}-1))$$
$$\text{where} K^{(n-1)} = 2^{n-((n-1)+1)} = 2^0 = 1$$
$$= \sum_{k=0}^{0} (2^{n-1})(2^{n-1-1})k + 2^{(n-1)-2}(2^{(n-1)-2} + 2^{(n-1)-1}) + 2^{(n-1)-2}(2^{n-3} + 2^{n-2} - 1)$$
$$= 2^{n-3}(2^1 - 1)(2^{n-3} + 2^{n-2} - 1 + 2^n) + 2^{(n-1)-2}(2^{n-3} + 2^{n-2} - 1). \tag{17}$$

Finally, the coefficient of $P(\{x_n\})$ is

$$\sum_{k=0} (2^{n-1})(2^n)k + (2^{n-1} + (2^{n-1}+1) + (2^{n-1}+2) + \cdots + (2^{n-1} + (2^{n-1}-1)$$
$$= \sum_{k=0} 2^n(2^{n-1})k + 2^{n-2}(2^{n-2} + 2^{n-1} - 1) \tag{18}$$
$$= 2^{n-2}(2^{n-2} + 2^{n-1} - 1).$$

Therefore by using (12)-(17), we have

$$\sum_{V=0}^{2^n-1} VP(V) = \sum_{j=1}^{n} \{ \sum_{k=0}^{2^{n-(j+1)} + 2^{n-(j+1)-1} + \cdots + 2^2 + 2^1 + 2^0} (2^{j-1})(2^j)k$$
$$+ (2^{j-1} + (2^{j-1}+1) + (2^{j-1}+2) + \cdots + (2^{j-1} + (2^{j-1}-1)\} p(\{x_j\})$$
$$= \sum_{j=1}^{n} \{(2^{j-2})(2^{n-j} - 1)(2^{j-2} + 2^{j-1} - 1 + 2^n) + 2^{j-2}(2^{j-2} + 2^{j-1} - 1)\} p(\{x_j\}). \tag{19}$$

6. Calculation of Statistical Quantities Based on Belief of Set

Using formulae from Hall and Knight book [3], we find sums of some finite series as per our requirement. Consider

$$\begin{aligned}\sum_{V=0}^{2^n-1} VBel(V) &= 0Bel(0)+1Bel(1)+2Bel(2)+3Bel(3)\\ &+4Bel(4)+5Bel(5)+6Bel(6)+7Bel(7)\\ &+8Bel(8)+9Bel(9)+10Bel(10)+11Bel(11)\\ &+12Bel(12)+13Bel(13)+14Bel(14)+15Bel(15)\\ &+\vdots\\ &+\cdots+(2^n-1)Bel(2^n-1)\\ &=\frac{1}{2^{n-1}}\{0\cdot 0+1\cdot 1p(\{x_1\})+2\cdot 1p(\{x_2\})+3\cdot 2(p(\{x_1\})+p(\{x_2\}))\\ &+4\cdot 1p(\{x_3\})+5\cdot 2(p(\{x_1\})+p(\{x_3\}))+6\cdot 2(p(\{x_2\})+p(\{x_3\}))\\ &+7\cdot 4(p(\{x_1\})+p(\{x_2\})+p(\{x_3\}))+8\cdot 1p(\{x_4\})+9\cdot 2(p(\{x_1\}+p(\{x_4\}))\\ &+10\cdot 2(p(\{x_2\}+p(\{x_4\}))+11\cdot 4(p(\{x_1\})+p(\{x_2\})+p(\{x_4\}))\\ &+12\cdot 2(p(\{x_3\})+p(\{x_4\}))+13\cdot 4(p(\{x_1\})+p(\{x_3\})+p(\{x_4\}))\\ &+14\cdot 4(p(\{x_2\}+p(\{x_3\})+p(\{x_4\}))\\ &+15\cdot 8(p(\{x_1\})+p(\{x_2\})+p(\{x_3\})+p(\{x_4\}))\\ &+\vdots\\ &+\cdots+(2^n-1)\cdot(2^{n-1})(p(\{x_1\}+p(\{x_2\}+\cdots+p(\{x_n\}))\end{aligned} \tag{20}$$

$$\begin{aligned}\sum_{V=0}^{2^n-1} VBel(V) &= \frac{1}{2^{n-1}}\{(1\cdot 1+3\cdot 2+5\cdot 2+7\cdot 4+\cdots+(2^n-1)\cdot(2^{n-1}))p(\{x_1\})\\ &+(2\cdot 1+3\cdot 2+6\cdot 2+7\cdot 4+10\cdot 2+11\cdot 4+14\cdot 4+15\cdot 8\\ &+\cdots+(2^n-2)\cdot(2^{n-2})+(2^n-1)\cdot(2^{n-1}))p(\{x_2\})\\ &+(4\cdot 1+5\cdot 2+6\cdot 2+7\cdot 4+12\cdot 2+13\cdot 4+14\cdot 4+15\cdot 8\\ &+\cdots+(2^n-4)(2^{n-3}+(2^n-3)(2^{n-2})\\ &+(2^n-2)(2^{n-2})+(2^n-1)(2^{n-1}))p(\{x_3\})\\ &+\vdots\end{aligned}$$

$$+ ((2^{n-1}) \cdot 1 + (2^{n-1} + 1) \cdot 2 + (2^{n-1} + 2) \cdot 2 + (2^{n-1} + 3) \cdot 4 + \cdots$$
$$+ (2^{n-1} - 4) \cdot (2^{n-3}) + (2^{n-1} - 3) \cdot (2^{n-2}) + (2^{n-1} - 2) \cdot (2^{n-2}) \tag{21}$$
$$+ (2^{n-1} - 1) \cdot (2^{n-1}))p(\{x_n\})\}.$$

To calculate coefficients of $P(\{x_j\}), j = 1, 2, \ldots, n$, we adopt following steps as:

1 Calculate recurrence relation between coefficients of $p(\{x_1\})$ with increasing values of n.

2 Calculate coefficient of $p(\{x_1\})$ with value of $n = j$.

3 Calculate difference of coefficients of consecutive probabilities i.e. $p(\{x_j\})$ and $p(\{x_{j+1}\})$ for fixed values of n.

4 Calculate coefficient of $p(\{x_j\})$ for given n.

6.1 Calculate recurrence relation between coefficients of $p(\{x_1\})$ with increasing values of n

Now consider $1 \cdot 1 + 3 \cdot 2 + 5 \cdot 2 + 7 \cdot 4 + \cdots + (2^n - 1) \cdot (2^{n-1})$. Here a_k represents coefficient of $p(\{x_1\})$ with value $n = k$. Here we use notation $a^r_{(1,Bel)}$ = coefficient of $p(\{x_1\})$ in $\sum VBel(V)$. Therefore for $\sum VBel(V)$ we have
$a^1_{(1,Bel)} = 1 \cdot 1 = 1$,
$a^1_{(2,Bel)} = 1 \cdot 1 + 3 \cdot 2 = 7$,
$a^1_{(3,Bel)} = 1 \cdot 1 + 3 \cdot 2 + 5 \cdot 2 + 7 \cdot 4 = 45$,
$a^1_{(4,Bel)} = 1 \cdot 1 + 3 \cdot 2 + 5 \cdot 2 + 7 \cdot 4 + 9 \cdot 2 + 11 \cdot 4 + 13 \cdot 4 + 15 \cdot 8 = 279$,
$a^1_{(5,Bel)} = 1701, \ldots$ and so on.
With analytic observations, we have, $a^1_{(1,Bel)} = 1 \cdot 1 = 1$,

$$\begin{aligned} a^1_{(2,Bel)} &= 1 \cdot 1 + 3 \cdot 2 \\ &= a^1_{(1,Bel)} + 3(2) \\ &= 3(a^1_{(1,Bel)}) + 4(1) \\ &= 3(a^1_{(1,Bel)}) + 2^2(3^0). \end{aligned} \tag{22}$$

$$\begin{aligned} a^1_{(3,Bel)} &= 1\cdot 1 + 3\cdot 2 + 5\cdot 2 + 7\cdot 4 \\ &= a^1_{(2,Bel)} + 5(2) + 7(4) \\ &= 3a^1_{(2,Bel)} + 8(3) \\ &= 3a^1_{(2,Bel)} + 2^3(3^1). \end{aligned} \tag{23}$$

$$\begin{aligned} a^1_{(4,Bel)} &= 1\cdot 1 + 3\cdot 2 + 5\cdot 2 + 7\cdot 4 + 9\cdot 2 + 11\cdot 4 + 13\cdot 4 + 15\cdot 8 \\ &= a^1_{(3,Bel)} + 9(2) + 11(4) + 13(4) + 15(8) \\ &= 3a^1_{(3,Bel)} + 16(9)) \\ &= 3a^1_{(3,Bel)} + 2^4(3^2)) \end{aligned} \tag{24}$$

Similarly, $a^1_{(5,Bel)} = 3a^1_{(4,Bel)} + 16(54) = 3a^1_{(4,Bel)} + 32(27) = 3a^1_{(4,Bel)} + 2^5(3^3), \dots$ and so on. Therefore we have recurrence relation as:

$$a^1_{(1,Bel)} = 1, \quad \text{and for } n = 2,3,4,\dots, \quad a^1_{(n,Bel)} = 2^n(3^{n-2}) + 3a^1_{(n-1,Bel)} \tag{25}$$

6.2 Calculate recurrence relation between coefficients of $p(\{x_1\})$ with value of n

by using (22)-(25)Consider

$$\begin{aligned} a^1_{(n,Bel)} &= 2^n(3^{n-2}) + 3a^1_{(n-1,Bel)} \\ &= 2^n(3^{n-2}) + 3(2^{n-1}3^{n-3} + 3a^1_{(n-2,Bel)}) \\ &= 2^n(3^{n-2}) + 2^{n-1}3^{n-2} + 3^2a^1_{(n-2,Bel)}) \\ &= 3^{n-2}(2^n + 2^{n-1}) + 3^2a^1_{(n-2,Bel)}) \\ &= 3^{n-2}(3(2^{n-1})) + 3^2a^1_{(n-2,Bel)}) \\ &= 3^{n-1}(2^{n-1}) + 3^2a^1_{(n-2,Bel)} \\ &\vdots \\ &= 3^{n-1}(2^{n-1}) + 3^{n-2}(2^{n-2} + 2^{n-3} + 2^{n-4} + \cdots + 2^2) + 3^{n-1}a^1_{(1,Bel)} \\ &= 3^{n-1}(2^{n-1}) + 3^{n-2}(\frac{2^{n-1} - 2^2}{2-1}) + 3^{n-1}a^1_{(1,Bel)} \\ &= 3^{n-1}(2^{n-1}) + 3^{n-2}(2^{n-1} - 1) \end{aligned} \tag{26}$$

6.3 Calculate difference of coefficients of consecutive probabilities i.e. $p(\{x_j\})$ and $p(\{x_{j+1}\})$ for fixed values of n.

For this, we calculate differences of order one vertically for coefficient of $p(\{x_1\})$ and $p(\{x_2\})$ for first four values of n i.e. $n = 2, 3, 4, 5$. Firstly we prepare a table of coefficients of $p(\{x_j\})$ for $j \leq n, \quad n = 1, 2, 3, 4, 5$ as:

n	$p(\{x_1\})$	$p(\{x_2\})$	$p(\{x_3\})$	$p(\{x_4\})$	$p(\{x_5\})$
1	1				
2	7	8			
3	45	48	54		
4	279	288	306	342	
5	1701	1728	1782	1890	2006

Here we use notation $d^r_{(p,q,j,Bel)}$ for difference between p^{th} and q^{th} columns with $p < q \leq n \& \quad j \geq q$ for j^{th} row in $\sum v^r Bel(v)$ for belief function Bel. For $j = q$, the difference $d^r_{(p,q,j,Bel)}$ is the first difference and further successive differences are dependent on this difference with increasing j.

Differences in $\sum v^1 Bel(v)$ vertically for coefficient of $p(\{x_1\})$ and $p(\{x_2\})$for first four values of n are 1, 3, 9, 27. In another way, $1 = 3^0, 3 = 3^1, 9 = 3^2, 27 = 3^4$. Therefore, the difference of coefficients $p(\{x_1\})$ and $p(\{x_2\})$ for $n = k$ is 3^{k-2}. Hence the difference of coefficients $p(\{x_1\})$ and $p(\{x_2\})$ for n is $d^1_{(1,2,n,Bel)} = 3^{n-2}$.

Differences in $\sum v^1 Bel(v)$ vertically for coefficient of $p(\{x_2\})$ and $p(\{x_3\})$for first three values of $n = 3, 4, 5$ are 6, 18, 54. In another way, $6 = 2(3) = 2(3^1), 18 = 2(9) = 2(3^2), 54 = 2(27) = 2(3^4)$. Therefore, the difference of coefficients $p(\{x_2\})$ and $p(\{x_3\})$ for $n = k$ is $2(3^{k-2})$. Hence the difference of coefficients $p(\{x_2\})$ and $p(\{x_3\})$ for n is $d^1_{(2,3,n,Bel)} = 2(3^{n-2})$.

Differences in $\sum v^1 Bel(v)$ vertically for coefficient of $p(\{x_3\})$ and $p(\{x_4\})$ for first two values of $n = 4, 5$ are 36, 108. In another way, $36 = 4(9) = 4(3^2), 108 = 4(27) = 4(3^4)$. Therefore, the difference of coefficients $p(\{x_3\})$ and $p(\{x_4\})$ for $n = k$ is $4(3^{k-2})$. Hence the difference of coefficients $p(\{x_3\})$ and $p(\{x_4\})$ for n is $d^1_{(3,4,n,Bel)} = 4(3^{n-2})$.

Differences of order one vertically for coefficient of $p(\{x_4\})$ and $p(\{x_5\})$for first value of $n = 5$ is 216. In another way, $216 = 8(27) = 8(3^4)$. Therefore, the difference of coefficients $p(\{x_4\})$ and $p(\{x_5\})$ for $n = k$ is $8(3^{k-2})$. Hence the difference of coefficients $p(\{x_4\})$ and $p(\{x_5\})$ for n is $d^1_{(4,5,n,Bel)} = 8(3^{n-2})$.

Continuing in this way, the difference in $\sum v^1 Bel(v)$, of coefficients $p(\{x_{j-1}\})$ and

$p(\{x_j\})$ for $n = k \geq j$ is $2^{j-2}(3^{k-2})$. Hence the difference of coefficients $p(\{x_4\})$ and $p(\{x_5\})$ for n is $d^1_{(j-1,j,Bel)} = 2^{j-2}(3^{n-2})$.
The difference of coefficients $p(\{x_1\})$ and $p(\{x_j\})$ for n is $d^1_{(1,j,n,Bel)} = 3^{n-2}(1+2+4+\cdots+2^{j-2})$ which on simplification is $3^{n-2}(2^{j-1}-1)$.

6.4 Calculate coefficient of $p(\{x_j\})$ for given n

The coefficient of $p(\{x_j\})$ for n is sum of coefficient of $p(\{x_1\})$ for n and difference of coefficients $p(\{x_1\})$ and $p(\{x_j\})$ for n. Therefore the coefficient of $p(\{x_j\})$ for n is $\frac{1}{2^{n-1}}\{3^{n-1}(2^{n-1}) + 3^{n-2}(2^{n-1}-1) + 3^{n-2}(2^{j-1}-1)\}$.
Therefore by using coefficient of $p(\{x_j\})$ for given n, we have

$$\begin{aligned} &\sum_{V-0}^{2^n-1} VBel(V) \\ &= \sum_{j=1}^{n}\{\frac{1}{2^{n-1}}\{3^{n-1}(2^{n-1}) + 3^{n-2}(2^{n-1}-1) + 3^{n-2}(2^{j-1}-1)\}p(\{x_j\}). \end{aligned} \tag{27}$$

7. Calculation of Statistical Quantities based on Plausibility of Set

As plausibility of set is an upper bound of probability of set, we will obtain upper limits of statistical quantities in this section. Now consider

$$\begin{aligned} &\sum_{V=0}^{2^n-1} VPl(V) = 0Pl(0) + 1Pl(1) + 2Pl(2) + 3Pl(3) \\ &+ 4Pl(4) + 5Pl(5) + 6Pl(6) + 7Pl(7) \\ &+ 8Pl(8) + 9Pl(9) + 10Pl(10) + 11Pl(11) \\ &+ 12Pl(12) + 13Pl(13) + 14Pl(14) + 15Pl(15) \\ &+ \\ &\vdots \\ &+ \cdots + (2^n-1)Pl(2^n-1) \\ &= \frac{1}{2^{n-1}}\{0 + 1\{2^{n-1}p(\{x_1\}) + 2^{n-2}[p(\{x_2\}) + p(\{x_3\}) + \cdots + p(\{x_n\})]\} \\ &+ 2\{2^{n-1}p(\{x_2\}) + 2^{n-2}[p(\{x_1\}) + p(\{x_3\}) + p(\{x_4\}) + \cdots + p(\{x_n\})]\} \\ &+ 3\{2^{n-1}(p(\{x_1\}) + p(\{x_2\})) + 3\cdot 2^{n-3}[p(\{x_3\}) + p(\{x_4\}) \end{aligned}$$

$$
\begin{aligned}
&+p(\{x_5\})+\cdots+p(\{x_n\})]\}\\
&+4\{2^{n-1}p(\{x_3\})+2^{n-2}[p(\{x_1\})+p(\{x_2\})+p(\{x_4\})+\cdots+p(\{x_n\})]\}\\
&+5\{2^{n-1}(p(\{x_1\})+p(\{x_3\}))+3\cdot 2^{n-3}[p(\{x_2\})+p(\{x_4\})\\
&+p(\{x_5\})+\cdots+p(\{x_n\})]\}\\
&+6\{2^{n-1}(p(\{x_2\})+p(\{x_3\}))+3\cdot 2^{n-3}[p(\{x_1\})+p(\{x_4\})\\
&+p(\{x_5\})+\cdots+p(\{x_n\})]\}\\
&+7\{2^{n-1}(p(\{x_1\})+p(\{x_2\})+p(\{x_3\}))+7\cdot 2^{n-4}[p(\{x_4\})\\
&+p(\{x_5\})+\cdots+p(\{x_n\})]\}\\
&+8\{2^{n-1}p(\{x_4\})+2^{n-2}[p(\{x_1\})+p(\{x_2\})+p(\{x_3\})+p(\{x_5\})\\
&+p(\{x_6\})+\cdots+p(\{x_n\})]\}\\
&+9\{2^{n-1}(p(\{x_1\})+p(\{x_4\}))+3\cdot 2^{n-3}[p(\{x_2\})+p(\{x_3\})+p(\{x_5\})+p(\{x_6\})\\
&+\cdots+p(\{x_n\})]\}\\
&+10\{2^{n-1}(p(\{x_2\})+p(\{x_4\}))+3\cdot 2^{n-3}[p(\{x_1\})+p(\{x_3\})+p(\{x_5\})+p(\{x_6\})\\
&+\cdots+p(\{x_n\})]\}\\
&+11\{2^{n-1}(p(\{x_1\})+p(\{x_2\})+p(\{x_4\}))+7\cdot 2^{n-4}[p(\{x_3\})+p(\{x_5\})+p(\{x_6\})\\
&+\cdots+p(\{x_n\})]\}\\
&+12\{2^{n-1}(p(\{x_3\})+p(\{x_4\}))+3\cdot 2^{n-3}[p(\{x_1\})+p(\{x_2\})+p(\{x_5\})+p(\{x_6\})\\
&+\cdots+p(\{x_n\})]\}\\
&+13\{2^{n-1}(p(\{x_1\})+p(\{x_3\})+p(\{x_4\}))+7\cdot 2^{n-4}[p(\{x_2\})+p(\{x_5\})+p(\{x_6\})\\
&+\cdots+p(\{x_n\})]\}\\
&+14\{2^{n-1}(p(\{x_2\})+p(\{x_3\})+p(\{x_4\}))+7\cdot 2^{n-4}[p(\{x_1\})+p(\{x_5\})+p(\{x_6\})\\
&+\cdots+p(\{x_n\})]\}\\
&+15\{2^{n-1}(p(\{x_1\})+p(\{x_2\})+p(\{x_3\})+p(\{x_4\}))+15\cdot 2^{n-5}[p(\{x_5\})+p(\{x_6\})\\
&+\cdots+p(\{x_n\})]\}\\
&+\\
&\vdots\\
&+(2^n-1)2^{n-1}(p(\{x_1\})+p(\{x_2\})+p(\{x_3\})+\cdots+p(\{x_n\}))\}
\end{aligned}
\tag{28}
$$

To calculate coefficients of $p(\{x_j\}), j = 1, 2, \ldots, n$, we adopt following steps as:

1 Calculate coefficient of $p(\{x_1\})$ for given n.

2 Calculate difference of coefficients of consecutive probabilities i.e. $p(\{x_j\})$ and $p(\{x_{j+1}\})$ for given n.

3 Calculate coefficient of $p(\{x_j\})$ for given n.

7.1 Calculation of coefficient of $p(\{x_1\})$ for given n

Here we classify sets according to cardinality of set containing some particular element $\{x_1\}$ and not containing $\{x_1\}$ viz. set containing $\{x_1\}$ having plausibility 2^{n-1}, singleton set not containing $\{x_1\}$ having plausibility $(2^1 - 1)2^{n-2}$, set of cardinality 2 but not containing $\{x_1\}$ having plausibility $(2^2 - 1)2^{n-3}$, set of cardinality 3 but not containing $\{x_1\}$ having plausibility $(2^3 - 1)2^{n-4}, \ldots$, set of cardinality r but not containing $\{x_1\}$ having plausibility $(2^r - 1)2^{n-(r+1)}$, and finally, set of cardinality $n-1$ but not containing $\{x_1\}$ having plausibility $2^{n-n}(2^{n-1} - 1)$.The coefficient of $p(\{x_1\})$ is

$$
\begin{aligned}
& \text{set containing } \{x_1\} \text{ having plausibility } 2^{n-1} \\
+ & \text{ singleton set not containing } \{x_1\} \text{ having plausibility } (2^1 - 1)2^{n-2} \\
+ & \text{ set of cardinality 2 but not containing } \{x_1\} \text{ having plausibility } (2^2 - 1)2^{n-3} \\
+ & \text{ set of cardinality 3 but not containing } \{x_1\} \text{ having plausibility } (2^3 - 1)2^{n-4} \\
\vdots & \\
+ & \text{ set of cardinality } r \text{ but not containing } \{x_1\} \text{ having plausibility } (2^r - 1)2^{n-(r+1)} \\
+ & \\
\vdots & \\
+ & \text{ set of cardinality } n-1 \text{ but not containing } \{x_1\} \text{ having plausibility } (2^{n-1} - 1)2^{n-n} \\
= & \, 2^{n-1}(1 + 3 + 5 + 7 + \cdots + (2^n - 1)) \\
+ & \, 2^{n-2} \cdot 1(2 + 4 + 8 + 16 + \cdots + 2^{n-1}) \\
+ & \, 2^{n-3} \cdot 3(6 + 10 + 12 + 18 + 20 + 24 + 34 + 36 + 40 + 48 + \cdots +
\end{aligned}
$$

$$
\begin{aligned}
&+(2^{n-1}+2)+(2^{n-1}+2+2)+(2^{n-1}+2+2+4)+\cdots+(2^{n-1}+2+2+4+8+\cdots+2^{n-3}))\\
&+2^{n-4}\cdot 7(14+22+26+28+38+42+44+50+52+56+\cdots)\\
&+2^{n-5}\cdot 15(30+46+54+58+60+78+86+90+92+102+106+108+114+116+120+\cdots)\\
&+2^{n-6}\cdot 31(62+94+110+118+122+124+\cdots)\\
&+2^{n-7}\cdot 63(126+\cdots)\\
&+\\
&\vdots\\
&+2^{n-(n-1)}(2^{n-2}-1)(2^{n-1}-2+\cdots+(2^{n}-(6+4))+(2^{n}-6))\\
&+2^{n-n}(2^{n-1}-1)(2^{n}-2)
\end{aligned} \tag{29}
$$

The coefficient of $p(\{x_1\})$ becomes

$$
\begin{aligned}
&= 2^{n-1}(1+3+5+7+\cdots+(2^{n}-1))\\
&+2^{n-2}\cdot 1(\sum_{i_1=1}^{n-1}2^{i_1})\\
&+2^{n-3}\cdot 3(\sum_{i_1=2}^{n-1}\sum_{i_2=1}^{i_1-1}2^{i_1}+2^{i_2})\\
&+2^{n-4}\cdot 7(\sum_{i_1=3}^{n-1}\sum_{i_2=2}^{i_1-1}\sum_{i_3=1}^{i_2-1}2^{i_1}+2^{i_2}+2^{i_3})\\
&+2^{n-5}\cdot 15(\sum_{i_1=4}^{n-1}\sum_{i_2=3}^{i_1-1}\sum_{i_3=2}^{i_2-1}\sum_{i_4=1}^{i_3-1}2^{i_1}+2^{i_2}+2^{i_3}+2^{i_4})\\
&+\\
&\vdots\\
&+2^{n-j}(2^{j-1}-1)(\sum_{i_1=j-1}^{n-1}\sum_{i_2=j-2}^{i_1-1}\sum_{i_3=j-3}^{i_2-1}\cdots\sum_{i_{j-1}=1}^{i_{j-2}-1}2^{i_1}+2^{i_2}+2^{i_3}+\cdots+2^{i_{j-1}})\\
&+\\
&\vdots
\end{aligned}
$$

$$+ 2^{n-(n-1)}(2^{n-2}-1)(\sum_{i_1=n-2}^{n-1}\sum_{i_2=n-3}^{i_1-1}\sum_{i_3=n-4}^{i_2-1}\cdots\sum_{i_{n-2}=1}^{i_{n-3}-1} 2^{i_1}+2^{i_2}+2^{i_3}+\cdots+2^{i_{n-2}})$$
$$+ 2^{n-n}(2^{n-1}-1)(\sum_{i_1=n-1}^{n-1}\sum_{i_2=n-2}^{i_1-1}\sum_{i_3=n-3}^{i_2-1}\cdots\sum_{i_{n-1}=1}^{i_{n-2}-1} 2^{i_1}+2^{i_2}+2^{i_3}+\cdots+2^{i_{n-1}}). \quad (30)$$

Now we will simplify summation part in each term separately and then simplify whole expression.

Summation Part in First Term:-

We have

$$1+3+5+7+\cdots+(2^n-1) = (2^{n-1})^2. \quad (31)$$

Summation Part in Second Term:-

We have

$$\sum_{i_1=1}^{n-1} 2^{i_1} = \frac{2^n-2}{2-1}. \quad (32)$$

Summation Part in Third Term:-

To simplify summation part in third term, we require following results based on Knight and Hall‘s book [6], as:

$$a+(a+d)r+(a+2d)r^2+(a+3d)r^3+\cdots+(a+(n-1)d)r^{n-1}$$
$$= \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)r^n}{1-r} \quad (33)$$

$$\text{Now } \sum_{k=1}^{n} k2^k = 2\sum_{k=1}^{n} k2^{k-1}$$
$$= 2[1+2\cdot 2^1+3\cdot 2^2+4\cdot 2^3+\cdots+n\cdot 2^{n-1}] \quad (34)$$

Comparing with above result, for series inside rectangular brakets, we get $a = d = 1, r = 2$.

$$\sum_{k=1}^{n} k2^k = 2[\frac{1}{1-2} + \frac{1(2)(1-2^{n-1})}{(1-2)^2} - \frac{(1+(n-1)1)2^n}{1-2}]$$
$$= 2[1+(n-1)2^n]. \quad (35)$$

The some part in summation part in third term in required expression becomes as:

$$\begin{aligned}\sum_{i_1=2}^{n-1} i_1 2^{i_1} &= \sum_{i_1=1}^{n-1} i_1 2^{i_1} - (1(2)) \\ &= 2[1+(n-2)2^{n-1}] - 2\end{aligned} \tag{36}$$

Therefore summation part in third term in required expression by using (33)-(36) becomes as:

$$\begin{aligned}\sum_{i_1=2}^{n-1}\sum_{i_2=1}^{i_1-1} 2^{i_1} + 2^{i_2} &= \sum_{i_1=2}^{n-1}\{(i_1-1)2^{i_1} + \sum_{i_2=1}^{i_1-1} 2^{i_2}\} \\ &= \sum_{i_1=2}^{n-1}(i_1-1)2^{i_1} + \frac{2^{i_1}-2}{2-1} \\ &= (n-2)(2^n-2)\end{aligned} \tag{37}$$

Summation Part in Fourth Term:-

To simplify summation part in fourth term, we require following results based on Knight and Hall‘s book [6], as:

Let $S = 1 + 3r + 5r^2 + \cdots + (2n-3)r^{n-2}$

$\Rightarrow \quad rS = r + 3r^2 + 5r^3 + \cdots + (2n-5)r^{n-2} + (2n-3)r^{n-1}$

$\Rightarrow \quad S(1-r) = 1 + 2r + 2r^2 + 2r^3 + \cdots + 2r^{n-2} - (2n-3)r^{n-1}$

$\Rightarrow \quad S = \frac{1}{1-r} + \frac{2r}{1-r}(\frac{1-r^{n-2}}{1-r}) - \frac{(2n-3)r^{n-1}}{(1-r)}.$

Using above sum, we have following sum as:

Let $S = a^2 + (a+d)^2 r + (a+2d)^2 r^2 + (a+3d)^2 r^3 + \cdots + (a+(n-1)d)^2 r^{n-1}$.

$\Rightarrow rS = a^2 r + (a+d)^2 r^2 + (a+2d)^2 r^3 + (a+3d)^2 r^4 + \cdots + (a+(n-1)d)^2 r^n$

On subtraction, we get

$$\begin{aligned}S(1-r) &= a^2 + 2adr + d^2 r + 2adr^2 + 3d^2 r^2 + 2adr^3 + 5d^2 r^3 + \cdots \\ &+ 2adr^{n-1} + [(n-1)^2 - (n-2)^2]d^2 r^{n-1} - (a+(n-1)d)^2 r^n \\ &= a^2 + 2adr(\frac{1-r^{n-1}}{1-r}) \\ &+ \frac{d^2 r}{1-r} + \frac{2d^2 r^2(1-r^{n-2})}{(1-r)^2} - \frac{(2n-3)d^2 r^n}{(1-r)} \\ &- (a+(n-1)d)^2 r^n\end{aligned} \tag{38}$$

$$\begin{aligned}
\therefore \quad S = &\frac{a^2}{1-r} + 2adr(\frac{1-r^{n-1}}{(1-r)^2}) \\
&+ \frac{d^2r}{(1-r)^2} + \frac{2d^2r^2(1-r^{n-2})}{(1-r)^3} - \frac{(2n-3)d^2r^n}{(1-r)^2} \\
&- \frac{(a+(n-1)d)^2r^n}{1-r}.
\end{aligned} \tag{39}$$

Put $a = d = 1, \quad r = 2,$

$$\begin{aligned}
S = &\frac{1^2}{1-2} + 2(1)(1)2(\frac{1-2^{n-1}}{(1-2)^2}) \\
&+ \frac{1^2(2)}{(1-2)^2} + \frac{2(1^2)(2^2)(1-2^{n-2})}{(1-2)^3} - \frac{(2n-3)1^2(2^n)}{(1-2)^2} \\
&- \frac{(1+(n-1)1)^2(2^n)}{1-2} \\
&= -3 + 2^n(n^2-2n+3)
\end{aligned} \tag{40}$$

Therefore summation part in fourth term in required expression by using (38)-(40) becomes as:

$$\begin{aligned}
&\sum_{i_1=3}^{n-1}\sum_{i_2=2}^{i_1-1}\sum_{i_3=1}^{i_2-1} 2^{i_1} + 2^{i_2} + 2^{i_3} \\
&= 2^{n-1}(n^2-4n+6) - (n-2)2^{n-1}) - 2^n - n^2 + n + 4n - 6 \\
&= (2^{n-1}-1)(n-2)(n-3).
\end{aligned} \tag{41}$$

Summation Part in Fifth Term:-

To simplify summation part in fifth term, we require following results based on Knight and Hall‘s book [6], as:

Let $S = a^3 + (a+d)^3r + (a+2d)^3r^2 + (a+3d)^3r^3 + \cdots + +(a+(n-1)d)^3r^{n-1}$.

$\Rightarrow \quad rS = a^3r(a+d)^3r^2 + (a+2d)^3r^3 + (a+3d)^3r^4 + \cdots + (a+(n-1)d)^3r^n.$

On substraction, we get

$$
\begin{aligned}
S - rS &= a^3 + 3a^2dr + 3ad^2r + d^3r + 3a^2dr^2 + 9ad^2r^2 + 7d^3r^2 + 3a^2dr^3 + 15ad^2r^3 + 19d^3r^3 \\
&+ \cdots + 3a^2dr^{n-1} + (3a(n-1)^2 - 3a(n-2)^2)d^2r^{n-1} + ((n-1)^3 - (n-2)^3)d^3r^{n-1} \\
&- (a + (n-1)d)^3r^n \\
&= a^3 + 3a^2dr(\frac{r^{n-1} - 1}{r - 1}) + 3ad^2r(\sum_2^n (2n-3)r^{n-2}) \\
&+ d^3r(\sum_2^n (3n^2 - 9n + 7)r^{n-2}) - (a + (n-1)d)^3r^n
\end{aligned} \tag{42}
$$

Put $a = d = 1$ and $r = 2$.

$$
\begin{aligned}
S(1-2) &= 1 + 3(1^2)(1)2(\frac{2^{n-1} - 1}{2 - 1}) + 3(1)(1^2)2(\sum_2^n (2n-3)2^{n-2}) \\
&+ 1^3 2(\sum_2^n (3n^2 - 9n + 7)2^{n-2}) - (1 + (n-1)1)^3 2^n
\end{aligned}
$$

$$
\begin{aligned}
S &= -1 - 6(2^{n-1} - 1) - 6(\sum_2^n (2n-3)2^{n-2}) \\
&- 2(\sum_2^n (3n^2 - 9n + 7)2^{n-2}) + n^3 2^n
\end{aligned} \tag{43}
$$

$$
S = -1 - 6(2^{n-1} - 1) - 3\sum_2^n n^2 2^{n-1} + 3\sum_2^n n2^{n-1} + \sum_2^n 2^n + n^3 2^n
$$

As we have formulae,
$\sum_1^n n^2 2^{n-1} = -3 + 2^n(n^2 - 2n + 3)$
and $\sum_1^n n2^n = 2[1 + (n-1)2^n]$
Using these formulae, we have

$$
\begin{aligned}
S &= -1 - 6(2^{n-1} - 1) - 3\sum_2^n n^2 2^{n-1} + 3\sum_2^n n2^{n-1} + \sum_2^n 2^n + n^3 2^n \\
&= 2^n(n^3 - 3(n^2) + 9n - 13) + 13.
\end{aligned} \tag{44}
$$

Therefore $\sum_1^n n^3 2^{n-1} = 2^n(n^3 - 3(n^2) + 9n - 13) + 13.$
$\Rightarrow \sum_1^n n^3 2^n = 2^{n+1}(n^3 - 3n^2 + 9n - 13) + 26.$
Therefore summation part in fifth term in required expression by using (42)-(44) becomes as:

$$\begin{aligned}
&\sum_{i_1=4}^{n-1}\sum_{i_2=2}^{i_1-1}\sum_{i_3=1}^{i_2-1}\sum_{i_4=1}^{i_3-1} 2^{i_1}+2^{i_2}+2^{i_3}+2^{i_4} \\
&=\sum_{i_1=4}^{n-1}\sum_{i_2=3}^{i_1-1}\sum_{i_3=2}^{i_2-1}\{(i_3-1)(2^{i_1}+2^{i_2}+2^{i_3})+\sum_{i_4=1}^{i_3-1}2^{i_4}\} \\
&=(1/6)(2^n-2)(n-2)(n-3)(n-4)
\end{aligned} \tag{45}$$

Therefore by using (31), (32), (37), (41) and (45), we have coefficient of $p(\{x_1\}$ in $\sum VPl(V)$ as:

$$\begin{aligned}
&2^{n-1}(1+3+5+7+\cdots+(2^n-1)) \\
&+2^{n-2}\cdot 1(\sum_{i_1=1}^{n-1}2^{i_1}) \\
&+2^{n-3}\cdot 3(\sum_{i_1=2}^{n-1}\sum_{i_2=1}^{i_1-1}2^{i_1}+2^{i_2}) \\
&+2^{n-4}\cdot 7(\sum_{i_1=3}^{n-1}\sum_{i_2=2}^{i_1-1}\sum_{i_3=1}^{i_2-1}2^{i_1}+2^{i_2}+2^{i_3}) \\
&+2^{n-5}\cdot 15(\sum_{i_1=4}^{n-1}\sum_{i_2=3}^{i_1-1}\sum_{i_3=2}^{i_2-1}\sum_{i_4=1}^{i_3-1}2^{i_1}+2^{i_2}+2^{i_3}+2^{i_4}) \\
&+ \\
&\vdots \\
&+2^{n-j}(2^{j-1}-1)(\sum_{i_1=j-1}^{n-1}\sum_{i_2=j-2}^{i_1-1}\sum_{i_3=j-3}^{i_2-1}\cdots\sum_{i_{j-1}=1}^{i_{j-2}-1}2^{i_1}+2^{i_2}+2^{i_3}+\cdots+2^{i_{j-1}}) \\
&+ \\
&\vdots \\
&+2^{n-(n-1)}(2^{n-2}-1)(\sum_{i_1=n-2}^{n-1}\sum_{i_2=n-3}^{i_1-1}\sum_{i_3=n-4}^{i_2-1}\cdots\sum_{i_{n-2}=1}^{i_{n-3}-1}2^{i_1}+2^{i_2}+2^{i_3})+\cdots+2^{i_{n-2}})
\end{aligned}$$

$$+ 2^{n-n}(2^{n-1}-1)(\sum_{i_1=n-1}^{n-1}\sum_{i_2=n-2}^{i_1-1}\sum_{i_3=n-3}^{i_2-1}\cdots\sum_{i_{n-1}=1}^{i_{n-2}-1}2^{i_1}+2^{i_2}+2^{i_3})+\cdots+2^{i_{n-1}})$$

$$= 2^{n-1}(2^{n-1})^2 \tag{46}$$

$$+\sum_{r=2}^{n}2^{n-r}(2^{r-1}-1)\frac{1}{(r-2)!}(2^n-2)(n-2)(n-3)(n-4)\cdots(n-(r-1)).$$

7.2 Calculate difference of coefficients of consecutive probabilities i.e. $p(\{x_j\})$ **and** $p(\{x_{j+1}\})$ **for given** n

We have table of coefficients in $\sum VPl(V)$ by neglecting denominator term 2^{n-1} and differences of consecutive columns in round brackets as:

n	$p(\{x_1\})$	$p(\{x_2\})$	$p(\{x_3\})$	$p(\{x_4\})$	$p(\{x_5\})$
1	1				
2	10 (1)	11			
3	94 (3)	97 (6)	103		
4	834 (9)	843 (18)	861 (36)	897	
5	7126 (27)	7153 (54)	7207 (108)	7315 (216)	7531

Therefore for n, differences of consecutive columns are

$$(2^0)(2^{n-2}),(2^1)(2^{n-2}),(2^2)(2^{n-2}),(2^3)(2^{n-2}),\cdots.$$

Hence for n, difference of $(j-1)^{th}$ and j^{th} columns is $(2^{j-2})(2^{n-2}) = 2^{n+j-4}$. Now difference of first and j^{th} columns is sum of differences of consecutive columns viz. first and second columns, second and third columns, third and fourth columns up to $(j-1)^{th}$ and j^{th} columns. Therefore difference of first and j^{th} columns is

$$\begin{aligned}&(2^0)(2^{n-2})+(2^1)(2^{n-2})+(2^2)(2^{n-2})+(2^3)(2^{n-2})+\cdots+(2^{j-2})(2^{n-2})\\&=(2^{n-2})(2^{j-1}-1)\end{aligned} \tag{47}$$

The coefficient of $p(\{x_j\})$ by neglecting denominator term 2^{n-1} is sum of coefficient of $p(\{x_1\})$ by neglecting denominator term 2^{n-1} and difference of first and j^{th} columns. The coefficient of $p(\{x_j\})$ by neglecting denominator term 2^{n-1} is

$$2^{n-1}(2^{n-1})^2+\sum_{r=2}^{n}2^{n-r}(2^{r-1}-1)\frac{1}{(r-2)!}(2^n-2)(n-2)(n-3)(n-4)\cdots(n-(r-1))$$
$$+(2^{n-2})(2^{j-1}-1) \tag{48}$$

Finally, the coefficient of $p(\{x_j\})$ by considering denominator term 2^{n-1} is

$$1/(2^{n-1})[2^{n-1}(2^{n-1})^2+\sum_{r=2}^{n}2^{n-r}(2^{r-1}-1)\frac{1}{(r-2)!}(2^n-2)(n-2)(n-3)(n-4)\cdots(n-(r-1))$$
$$+(2^{n-2})(2^{j-1}-1)] \tag{49}$$

Therefore by using (49), we have

$$\sum_{V=0}^{2^n-1}=\sum_{j=1}^{n}1/(2^{n-1})[2^{n-1}(2^{n-1})^2+\sum_{r=2}^{n}2^{n-r}(2^{r-1}-1)\frac{1}{(r-2)!}(2^n-2)(n-2)(n-3)(n-4)$$
$$\cdots(n-(r-1))+(2^{n-2})(2^{j-1}-1)] \tag{50}$$

Therefore lower limits and upper limits of arithmetic mean is obtained by using equations (9), (19), (27) and (50).

8. Illustrative Example

Let $X\sim Binomial(n,p)$. Therefore $p(x)=\binom{n}{p}p^xq^{n-x}$. Now we connsider $n=4, p=2/3$ and $q=1-p=1/3$. The distribution of X is

X	:	0	1	2	3	4	$Total$
$p(x)$	:	1/81	8/81	24/81	32/81	16/81	1

Now by notation

$$p(v)=p(A_v)\quad v=0,1,2,3,\cdots,2^5-1$$

By indexing of sets, $F(x)=P(X\leq x)=p(\{0,1,2,3,\ldots,x\})=p(A_v)$ and only in this case, relation between x and v is $v=2^{x+1}-1,\quad x=0,1,2,3,4$. By lower and upper limits of probability of sets, $Bel(A_v)\leq P(A_v)\leq Pl(A_v)$, we get $Bel(A_v)\leq F(X)\leq x)\leq Pl(v),\quad x=0,1,2,3,4$ and$v=2^{x+1}-1$ Therefore we get lower and upper limits of distribution function of given probability distribution including the case of subset $\emptyset$ as

Sr. No.	x	subset of Θ	v	$Bel(A_v)$	$P(A_v)$	$Pl(A_v)$
1	0	$\{x_0\}$	1	1/1296	1/81	656/1296
2	1	$\{x_0, x_1\}$	3	18/1296	9/81	1008/1296
3	2	$\{x_0, x_1, x_2\}$	7	132/1296	33/81	1200/1296
4	3	$\{x_0, x_1, x_2, x_3\}$	15	520/1296	65/81	1280/1296
5	4	$\{x_0, x_1, x_2, x_3, x_4\}$	31	1296/1296=1	81/81=1	1296/1296=1

Now we will calculate mean of given probability distribution in following table.

X	$P(X)$	$XP(X)$
0	1/81	0
1	8/81	8/81
2	24/81	48/81
3	32/81	96/81
4	16/81	64/81
$\sum$	1	216/81

Therefore we have mean of given probability distribution $\mu_1^{'} = 216/81$. Now we will calculate $\sum_{V=0}^{2^n-1} VBel(V)$, $\sum_{V=0}^{2^n-1} VP(V)$ and $\sum_{V=0}^{2^n-1} VPl(V)$ from coefficients of $P(\{x_j\}), j = 0, 1, 2, 3, 4$. These values are given in following tables.

$\sum$	$P(\{x_0\})$	$P(\{x_1\})$	$P(\{x_2\})$	$P(\{x_3\})$	$P(\{x_4\})$
$\sum V^1 Bel(V)$	1701	1728	1755	1782	1809
$\sum V^1 P(V)$	256	264	280	312	376
$\sum V^1 Pl(V)$	7126	7153	7207	7315	7531

First Raw Moments based on Probability of set :-

The r^{th} ordered raw moment based on probability of set is denoted by $\mu_r^{''}$ and is given by $\mu_r^{''} = \sum_{v=0}^{2^5-1} v^r p(v)$. Now we calculate first raw moments based on probability of set viz. $\mu_1^{''}$.

$$\begin{aligned} \mu_1^{"} &= \sum_{v=0}^{2^5-1} vp(v) \\ &= 309.728395. \end{aligned} \tag{51}$$

First Raw Moments based on Belief of set :-

The r^{th} ordered raw moment based on belief of set is denoted by $\underline{\mu}_r^{''}$ and is given by $\underline{\mu}_r^{''} = \sum_{v=0}^{2^5-1} v^r Bel(v)$. Now we calculate first raw moments based on belief of set viz. $\underline{\mu}_1^{''}$.

$$\begin{aligned}\underline{\mu}_1^{''} &= \sum_{v=0}^{2^5-1} vBel(v)\\ &= 0Bel(0) + 1Bel(1) + 2Bel(2) + 3Bel(3) + \cdots + (2^5-1)Bel(2^5-1) = 117.645833.\end{aligned} \tag{52}$$

First Raw Moments based on Plausibility of set :-

The r^{th} ordered raw moment based on plausibility of set is denoted by $\overline{\mu}_r^{''}$ and is given by $\overline{\mu}_r^{''} = \sum_{v=0}^{2^5-1} v^r Pl(v)$. Now we calculate first raw moments based on plausibility of set viz. $\overline{\mu}_1^{''}$.

$$\begin{aligned}\overline{\mu}_1^{''} &= \sum_{v=0}^{2^5-1} vPl(v)\\ &= 0Pl(0) + 1Pl(1) + 2Pl(2) + 3Pl(3) + \cdots + (2^5-1)Pl(2^5-1) = 456.708333.\end{aligned} \tag{53}$$

Lower and Upper limits of First Raw Moment i.e. Arithmetic Mean :-

By using (9) and (51)-(53), we calculate lower and upper limits of first raw moment,

$$\begin{aligned}\frac{\underline{\mu}_1^{''}}{\mu_1^{''}}\mu_1^{'} \leq \mu_1^{'} &\leq \frac{\overline{\mu}_1^{''}}{\mu_1^{''}}\mu_1^{'}\\ 1.012895 \leq 2.6667 &\leq 3.932119\\ \underline{\mu}_1^{'} \leq \mu_1^{'} &\leq \overline{\mu}_1^{'}.\end{aligned} \tag{54}$$

8. Future Scope

Using same approach, we can obtain lower and upper limits of several statistical quantities such as k^{th} ordered raw and central moments, coefficients of skewness and kurtosis based on central moments for probability distributions of one variable. Same approach can be extended to multivariate probability distributions.

9. Conclusion

Instead of having single value of statistical quantity, it is always better to have an interval in which this required single value is included. In this way we include uncertainty regarding a single value of statistical quantity. While obtaining lower and upper limits of distribution function and mean of probability distribution, we have taken care of getting more suitable limits. The lower and upper limits obtained by this approach are too wide but definitely contains all possibilities (i.e. all cases of uncertianty). This is one of the approach to find lower and upper limits of statistical quantities. We hope that this approach may be very useful in further research.

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