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A NOTE ON BETA FUNCTION AND FOURIER TRANSFORM

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Abstract

In this paper, we use the Fourier Transform and convolution theorem of Fourier Transform to prove the relation between Gamma function and beta function. It is possible that this technique of proof may be applied to solve the other problems involving beta function.

1. Introduction

The beta function [1], also called the Euler integral of the first kind, is a special function defined by $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$,for $\text{Re}(p) > 0$, $\text{Re}(q) > 0$.
The gamma function [2], also called the Euler integral of the second kind, is defined as convergent improper integral $\Gamma(n) = \int\limits_0^\infty e^{-x}x^{n-1}dx$ for $\text{Re}(n) > 0$.
Properties of beta function:

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1. $B(p,q) = B(q,p)$
2. $B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx$, Re$(p) > 0$, Re$(q) > 0$
3. $B(p,q) = 2\int_0^{\frac{\pi}{2}} sin\theta^{2p-1} cos\theta^{2q-1} dx$, Re$(p) > 0$, Re$(q) > 0$

The key property of beta function is $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$,which is to be proved in main results by using convolution theorem of Fourier transform.

Identities of beta and gamma function:

1. $B(p,q) = B(p+1,q) + B(p,q+1)$, $p,q > 0$
2. $B(p,q+1) = \frac{q}{p} B(p+1,q) = \frac{q}{p+q} B(p,q)$, $p,q > 0$
3. $\Gamma(p+1) = p\Gamma(p)$, $p > 0$

The proof of the relation between Gamma function and Beta function requires the following definitions and results.

Definition 1.1 [5] : The Fourier transform of the complex valued function $f(x)$ is defined by the integral, $F[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$, provided that integral exists for complex parameter s, and the Inverse Fourier transform of $F(s)$ is defined by the integral $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$.

Definition 1.2 [4] : Convolution: The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as $f(x) * g(x) = \int_{-\infty}^{\infty} f(u) g(x-u) du$

Lemma 1.3 [5] : Convolution Theorem : The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms, that is $F[f(x) * g(x)] = F[\int_{-\infty}^{\infty} f(u) g(x-u) du] = F[f(x)] F[g(x)]$.

Definition 1.4 [5] : Heaviside's unit function: The Heaviside's unit step function u(t-a) is defined as follows:

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geqslant a \end{cases} \tag{1}$$

The product

$$f(t)u(t-a) = \begin{cases} 0 & t < a \\ f(t) & t \geqslant a \end{cases} \tag{2}$$

2. Main Results

Theorem 2.1 : Relation between Gamma function and Fourier transform:If $s > 0$, then $F[u(t)t^{n-1}] = i^n \frac{\Gamma(n)}{s^n}$, where $u(t)$ is the unit step function and $n > 0$.

Proof :

$$Consider, F[u(t)t^{n-1}] = \int_{-\infty}^{\infty} u(t)t^{n-1}e^{ist}dt \tag{3}$$

$$= \int_{-\infty}^{0} u(t)t^{n-1}e^{ist}dt + \int_{0}^{\infty} u(t)t^{n-1}e^{ist}dt \tag{4}$$

$$= \int_{0}^{\infty} t^{n-1}e^{ist}dt \tag{5}$$

put $ist = -x$, $t = \frac{-x}{is}$, $dt = \frac{-dx}{is}$

$$F[u(t)t^{n-1}] = \int_{0}^{\infty} (\frac{-x}{is})^{n-1}e^{-x}(\frac{-dx}{is}) \tag{6}$$

$$= i^n \frac{1}{s^n} \int_{0}^{\infty} e^{-x}x^{n-1}dx \tag{7}$$

$$= i^n \frac{\Gamma(n)}{s^n} \tag{8}$$

□

Theorem 2.2 : If $m > 0$, $n > 0$, then $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Proof : Let $F[u(t)t^{m-1}] = i^m \frac{\Gamma(m)}{s^m}$, $F[u(t)t^{n-1}] = i^n \frac{\Gamma(n)}{s^n}$

We have, by convolution theorem,

$$F[f(t)]F[g(t)] = F[f(t) * g(t)] = F[\int_{-\infty}^{\infty} f(y)g(t-y)dy] \tag{9}$$

therefore

$$F[u(t)t^{m-1}]F[u(t)t^{n-1}] = F[\int_{-\infty}^{\infty} u(y)y^{m-1}u(t-y)(t-y)^{n-1}dy] \tag{10}$$

that is,

$$i^m \frac{\Gamma(m)}{s^m} i^n \frac{\Gamma(n)}{s^n} = F[\int_{-\infty}^{\infty} u(y)y^{m-1}u(t-y)(t-y)^{n-1}dy] \tag{11}$$

Let $h(t) = \int_{-\infty}^{\infty} u(y)y^{m-1}u(t-y)(t-y)^{n-1}dy$

Since,

$$u(y)u(t-y) = \begin{cases} 1 & 0 < y < t, t \geqslant 0 \\ 0 & Otherwise \end{cases} \quad [3] \tag{12}$$

$$h(t) = \int_0^t y^{m-1}(t-y)^{n-1} dy \tag{13}$$

that is,

$$h(t) = \begin{cases} \int_0^t y^{m-1}(t-y)^{n-1} dy & t \geqslant 0 \\ 0 & t < 0 \end{cases} \tag{14}$$

or

$$h(t) = u(t) \int_0^t y^{m-1}(t-y)^{n-1} dy, \quad [3] \tag{15}$$

where

$$u(t) = \begin{cases} 1 & t \geqslant 0 \\ 0 & t < 0 \end{cases} \tag{16}$$

therefore , equation (11) becomes,

$$i^{m+n} \frac{\Gamma(m)\Gamma(n)}{s^{m+n}} = F[u(t) \int_0^t y^{m-1}(t-y)^{n-1} dy] \tag{17}$$

put, $y = tx$, $dy = tdx$, (new limits are x=0, x=1),

$$i^{m+n} \frac{\Gamma(m)\Gamma(n)}{s^{m+n}} = F[u(t) \int_0^1 (tx)^{m-1}(t-tx)^{n-1} tdx] \tag{18}$$

$$= F[u(t)t^{m+n-1}] \int_0^1 x^{m-1}(1-x)^{n-1} dx \tag{19}$$

$$= i^{m+n} \frac{\Gamma(m+n)}{s^{m+n}} B(m,n) \tag{20}$$

Therefore,

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n) \tag{21}$$

□

3. Conclusion

The Fourier transform technique can be used to solve problems involving Beta function instead of probability theory.

References

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