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NEW CONSONANT BELIEF FUNCTION INDUCED BY PROBABILITY MASS FUNCTION

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Abstract

Usually authors are using approaches from belief function to probability and rarely some of the authors are using approaches from probability to belief functions. In this paper, we define a consonant basic belief assignment induced by probability mass function hence belief function. Here we will study various properties of consonant basic belief assignment induced by probability density function defined by us. Here we obtain lower and upper limits of raw and central moments, and coefficients of skewness and kurtosis of discrete probability distributions.

1. Introduction

In the world of uncertainty, each and every incidence occurring in our day to day life always follows some known or unknown probability distribution. Therefore choice of appropriate probability distribution plays an important role in decision making. Hence it becomes necessary that we should know common characteristics of all probability distributions.

Key Words : *Belief function, Consonant basic belief assignment, Plausibility function, Probability mass function.*

Here we define a new transformation which transforms probability mass function into basic belief assignment hence belief function for consonant focal elements. While obtaining this new transformation, we concentrate on sufficient axioms for basic belief assignment which must be satisfied by our new transformation. Once we have obtained such new transformation, we are able to find other functions related to basic belief assignment. Also we will check that this new transformation satisfies some more additional properties so that we can recognize the true class of this new transformation.
In this paper, firstly we summrise preliminaries of discrete belief functions and probability functions then in section 3, we will explain steps in the development of this new transformation. Nextly We deduce some results of our discrete belief function. In section 4, we obtain lower and upper limits of raw and central moments, and coefficients of skewness and kurtosis. In section 5, we illustrate these concepts by an example. Finally, we conclude about this paper and give list of references.
Now we summarize preliminaries of discrete belief functions and probability functions.

2. Preliminaries

2.1 Discrete Belief Function Theory

Frame of Discernment : Dictionary meaning of Frame of Discernment is frame of good judgement insight. The word discern means recognize or find out or hear with difficulty. In Shafer's book [7], if frame of discernment Θ is

$$\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$$

then every element of Θ is a proposition. The propositions of interest are in one -to -one correspondence with the subsets of Θ. The set of all propositions of interest corresponds to the set of all subsets of Θ, denoted by 2^Θ.
If Θ is *frame of discernment*, then a function $m : 2^\Theta \to [0, 1]$ is called **basic probability assignment** whenever $m(\emptyset) = 0$ and $\sum_{A\subset\Theta} m(A) = 1$. The quantity $m(A)$ is called A's **basic probability number** and it is a measure of the belief committed exactly to A.The *total belief* committed to A is sum of $m(B)$, for all proper subsets B of A. . A function $Bel : 2^\Theta \to [0, 1]$ is called **belief function** over Θ if it satisfies $Bel(A) = \sum_{B\subset A} m(B)$. If Θ is a frame of discernment, then a function $Bel : 2^\Theta \to [0, 1]$ is *belief function* if and only if it satisfies following conditions

1. $Bel(\emptyset) = 0$.

2. $Bel(\Theta) = 1$.

3. For every positive integer n and every collection $A_1, A_2, \ldots, A_n$ of subsets of Θ

$$Bel(A_1 \cup A_2 \cup \ldots \cup A_n) \geq \sum_{I \subset \{1,2,\cdots,n\}} (-1)^{|I|+1} Bel(\bigcap_{i \in I} A_i). \quad (1)$$

A subset of a frame Θ is called a **focal element** of a belief function Bel over Θ if $m(A) > 0$. The union of all the focal elements of a belief function is called its **core**.The quantity $Q(A) = \sum_{B \subset \Theta, A \subset B} m(B)$ is called **commonality number** for A which measures the total probability mass that can move freely to every point of A. A function $Q : 2^{\Theta} \to [0, 1]]$ is called **commonality function** for Bel. Also $Bel(A) = \sum_{B \subset \bar{A}}$ and $Q(A) = \sum_{B \subset A} (-1)^{|B|} Bel(\bar{B})$for all $A \subset \Theta$.

Degree of doubt :

$$Dou(A) = Bel(\bar{A}) \text{ or } Bel(A) = Dou(\bar{A}) \text{ and } pl(A) = 1 - Dou(A) = \sum_{A \cap B \neq \emptyset} m(B) \quad (2)$$

which expresses the extent to which one finds A credible or plausible. In Dempster‘s articles [4, 5], we have relation between belief function, probability function and plausibility function is

$$Bel(A) \leq p(A) \leq Pl(A), \quad \forall A \subset \Theta. \quad (3)$$

In Billingsley [2], a function $P : \Theta \to [0, 1]$ is called **probability function** if

1 $\forall A \in \Theta, \quad 0 \leq P(A) \leq 1$.

2 $P(\Theta) = 1$.

A set function μ on a frame of discernment Θ is a **measure** if it satisfies following three conditions:

1. $\mu(A) \in [0, \infty], \quad$ for all $A \in \Theta$.

2. $\mu(\emptyset) = 0$.

3. **Additive Property : For collection** $A_1, A_2, \ldots, A_n, \ldots$,

$$Bel(\cup_{i=1}^{\infty} A_i) = \sum_{\substack{I \subset 1,2,\ldots,n,\ldots \\ I \neq \emptyset}} (-1)^{|A|+1} Bel(\cap_{i=1}^{\infty} A_i). \quad (4)$$

The measure is finite or infinite as $\mu(\Theta) < \infty$ or $\mu(\Theta) = \infty$. It is probability measure if $\mu(\Theta) = 1$ [2].

In Shafer‘s book [7], If $\Theta$is frame of discernment then a function $Bel : 2^{\Theta} \to [0, 1]$ is called **Bayesian Belief Function** if

1 $Bel(\emptyset) = 0$,

2 $Bel(\Theta) = 1$,

3 $Bel(A \cup B) = Bel(A) + Bel(B)$ whenever $A, B \in \Theta$ and $A \cap B = \emptyset$.

Also we have some other basic belief assignments and we will briefly introduce these *bba*s.

Classical Pignistic Probability:- Philippe Smets [9] had given basic idea about classical pignistic probability and implemented in Transferable Belief Model. It transfers positive mass of belief of each non-specific element onto the singletons involved in that element split by the cardinality of the proposition when working with normalized basic belief assignments. In TBM, the classical pignistic probability is
$BetP(\emptyset) = 0$ and $\forall \quad A \in 2^{\Theta} - \{\emptyset\}$

$$BetP(A) = \sum_{\substack{B \in 2^{\Theta}, \\ B \neq \emptyset}} \frac{|A \cap B|}{|B|} \frac{m(B)}{1 - m(\emptyset)}. \tag{5}$$

In Shafer [7], $m(\emptyset) = 0$ hence above formula becomes

$$BetP(\theta_i) = m\theta_i + \sum_{\substack{B \in 2^{\Theta}, \\ \theta_i \subset B}} \frac{m(B)}{|B|} \text{ and } BetP(A) = \sum_{\theta_i \in A} BetP(\theta_i). \tag{6}$$

John Sudano [10, 11, 12], had developed transformations which approximates quantative belief mass m by subjective probabilities viz. Transformation Proportional to Plausibility, Transformation Proportional to Normalized Plausibility, Transformation Proportional to all Plausibilities, Hybrid Pignistic Probability and Probability Information Content. In [3], Cuzzolin developed transformation $CuzzP(.)$ for any finite and discrete frame of discernment $\Theta \quad n \geq 2$, satisfying Shafer's model as

$$CuzzP(\theta_i) = m(\theta_i) + \frac{\triangle(\theta_i)}{\sum_{j=1}^{n} \triangle(\theta_j)} \times TNSM \tag{7}$$

with $\triangle(\theta_i) = Pl(\theta_i) - m(\theta_i)$ and

$$TNSM = 1 - \sum_{j=1}^{n} m(\theta_j) = \sum_{A \in 2^{\Theta}, |A|>1} m(A). \tag{8}$$

Shannon [8] developed transformation discrete probability measure $H(.)$ for discrete frame of discernment Θ by

$$H(P) = -\sum_{i=1}^{n} P(\{\theta_i\}) log_2 P(\{\theta_i\}) \tag{9}$$

Normalized Shannon entropy is dual of PIC metric.

We have some results about interval arithmatic from Moore's book [6] as:

Let $X = [\underline{X}, \overline{X}]$ and $Y = [\underline{Y}, \overline{Y}]$ be any intervals, in set of real numbers. Here $\underline{X} = min.\{x : x \in X\}$ and $\overline{X} = max.\{x : x \in X\}$. Therefore $\underline{X}$ and $\overline{X}$ are lower and upper limits of X respectively. The computations with intervals are as:

$$X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}] \tag{10}$$

$$X - Y = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}] \tag{11}$$

$$X \cdot Y = [MinS, MaxS], \quad \text{Where } S = \{\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}\}. \tag{12}$$

$$X/Y = \begin{cases} [\underline{X}/\underline{Y}, \infty] & \text{if } \overline{X} \leq 0 \text{ and } \overline{Y} = 0 \\ [-\infty, \overline{X}/\overline{Y}] \cup [\overline{X}/\overline{Y}, \infty] & \text{if } \overline{X} \leq 0 \text{ and } \underline{Y} < 0 < \overline{Y} \\ [-\infty, \overline{X}/\overline{Y}] & \text{if } \overline{X} \leq 0 \text{ and } \underline{Y} = 0 \\ [-\infty, \infty] & \text{if } \underline{X} < 0 < \overline{X} \\ [-\infty, \underline{X}/\underline{Y}] & \text{if } \underline{X} \geq 0 \text{ and } \overline{Y} = 0 \\ [-\infty, \underline{X}/\underline{Y}] \cup [\underline{X}/\overline{Y}, \infty] & \text{if } \underline{X} \geq 0 \text{ and } \underline{Y} < 0 < \overline{Y} \\ [\underline{X}/\overline{Y}] & \text{if } \underline{X} \geq 0 \text{ and } \underline{Y} = 0 \end{cases} \tag{13}$$

Also $f(X) = \{f(x) : x \in X\}$. It is always beneficial to work on separate intervals instead of their unions and draw conclusions.

The necessary information about probability mass function, distribution function, raw moments, central moments and coefficients of skewness and kurtosis, is refered from Bansi Lal and Sanjay Arora book [1].

3. New Consonant Basic Belief Assignment

Let $p(x)$ and $P(X \leq x)$ be probability density function and distribution function of probability distribution under study respectively. We know that differentiation of distribution function $P(X \leq x)$, is a probability density function $p(x)$. If some subset $A = \{a_i, a_j, a_k\}$ of $\Theta = \{a_1, a_2, \ldots, a_n\}$ is of our interest. WOLOG, assume that A is subset of Θ and whose probability is distribution function for some $x = s$ i.e $p(A) = P(X \leq s)$. Here we have to concentrate on important condition that i, j and k are in some order i.e. starting values of X is i, all are in some order and all are less than n. If this condition is not satisfied then we have to search for another probability distribution in which i, j and k are successive in some order. Therefore firstly the subset A should be chosen and according to order of occurrence of it's elements, we have to search for probability distribution in which i, j and k are in some order.

For consonant *bba*, it's focal elements are nested. Without loss of generality, we assume following embedding of focal elements satisfy our requirement :

$$\{a_1\} \subseteq \{a_1, a_2\} \subseteq \{a_1, a_2, a_3\} \subseteq \cdots \subseteq \{a_1, a_2, \ldots, a_r\} \subseteq \{a_1, a_2, \ldots, a_n\} = \Theta \tag{14}$$

We have transformation for consonant *bba*,

$$m(A) = \frac{p(A)}{K}, \tag{15}$$

where $K = n * p(\{a_1\}) + (n-1) * p(\{a_2\}) + (n-2) * p(\{a_3\}) + \cdots + 2 * p(\{a_{n-1}\}) + p(\{a_n\})$. Note that $m(\emptyset) = 0$ as $p(\emptyset) = 0$. Also

$$\sum_{A \subseteq \Theta} A = \frac{p(\{a_1\}) + p(\{a_1, a_2\}) + p(\{a_1, a_2, a_3\}) + \cdots + p(\{a_1, a_2, \ldots, a_n\})}{K}$$
$$= \frac{K}{K} = 1.$$

Therefore m defined as above is a consonant basic belief assignment . We have following functions related to this basic belief assignment as:

3.1 Belief Function

If $|A| = r$ and $|\Theta| = n$ then number of subsets of A containing $\{a_1\}$ are r, containing $\{a_1, a_2\}$ are $r - 1$, containing $\{a_1, a_2, a_3\}$ are $r - 2$, …, containing $\{a_1, a_2, \ldots, a_k\}$ are $r - (k + 1)$, …, containing $\{a_1, a_2, \ldots, a_{r-1}\}$ are 2, containing $\{a_1, a_2, \ldots, a_r\}$ is 1. The belief function of set $A = \{a_1, a_2, \ldots, a_r\}$ is

$$\begin{aligned} Bel(A) &= \sum_{B \subseteq A} m(B) \\ &= m(\{a_1\}) + m(\{a_1, a_2\}) + m(\{a_1, a_2, a_3\}) + \cdots + m(\{a_1, a_2, \ldots, a_r\}) \\ &= \frac{1}{K} \sum_{\substack{\{a_i\} \in A; \\ i=0}}^{r-1} (r-i) p(\{a_{i+1}\}) \end{aligned} \quad (16)$$

3.2 Commonality Function

If $|A| = r$ and $|\Theta| = n$, then the number of subsets of Θ containing A are such that
$A = \{a_1, a_2, \ldots, a_r\}$ is contained in $n - r + 1$ subsets of Θ
$\{a_1, a_2, \ldots, a_r, a_{r+1}\}$ is contained in $n - r$ subsets of Θ
$\{a_1, a_2, \ldots, a_r, a_{r+1}, a_{r+2}\}$ is contained in $n - r$ subsets of Θ
$\vdots$
$\{a_1, a_2, \ldots, a_r, a_{r+1}, \ldots, a_{n-1}\}$ is contained in 2 subsets of Θ
$\{a_1, a_2, \ldots, a_r, a_{r+1}, \ldots, a_{n-1}, a_n\}$ is contained in 1 (i.e. only one subset Θ) subset of Θ.

From result : $p(\{a_1, a_2, \ldots, a_r\}) = p(\{a_1\}) + p(\{a_2\}) + \cdots + p(\{a_r\})$ since singletons have empty intersections. By applying this result to embedding given above, we get
$\{a_1\}, \{a_2\}, \ldots, \{a_r\}$ are repeated $n - (r-1)$ times.
$\{a_{r+1}\}$ is repeated $n - r$ times.
$\{a_{r+2}\}$ is repeated $n - (r+1)$ times.
$\{a_{r+3}\}$ is repeated $n - (r+2)$ times.
$\vdots$
$\{a_{n-1}\}$ is repeated 2 times.
$\{a_n\}$ is repeated only one time.
Therefore formula for commonality function becomes,

$$\begin{aligned} Q(A) &= \sum_{B \supseteq A} m(B) \\ &= m(\{a_1, a_2, \ldots, a_r\}) + m(\{a_1, a_2, \ldots, a_r, a_{r+1}\}) + m(\{a_1, a_2, \ldots, a_r, a_{r+1}, a_{r+2}\}) \\ &+ \cdots + m(\{a_1, a_2, \ldots, a_r, a_{r+1}, a_{r+2}, \ldots, a_n\}) \end{aligned}$$

$$= \frac{1}{K}\{(n-(r-1))\sum_{\{a_i\}\in A, i=1}^{r} p(\{a_i\}) + \sum_{\substack{\{a_i\}\notin A,\\ i=r+1}}^{n} (n-i-1)p(\{a_i\})\} \tag{17}$$

3.3 Plausibility Function

By embedding considered above, we observe that : for any subset A, subsets of Θ are either subsets of A or supersets of A therefore they have non-empty intersections i.e. for any two subsets of Θ, except $\emptyset$, have non-empty intersections hence every subsets of Θ, except $\emptyset$, have non-empty intersection with A. Therefore formula for plausibility function becomes

$$\begin{aligned} Pl(A) &= m(\{a_1\}) + m(\{a_1,a_2\}) + m(\{a_1,a_2,a_3\}) + \cdots + m(\{a_1,a_2,a_3,\ldots,a_n\}) \\ &= 1. \end{aligned} \tag{18}$$

4. Lower and Upper Limits of Moments

In this section, we calculate upper and lower limits of raw and central moments based on probability of set, using (3) : belief and plausibility functions are lower and upper limits of probability respectively.

4.1 Raw moments based on probability of set

Here we use $p(v) = p(A_v) = p(\{a_1,a_2,\ldots,a_v\}), \quad v = 1,2,\ldots,n$ and $p(0) = p(\emptyset) = 0$. Therefore we get r^{th} ordered raw moments based on probability of set as:

$$\begin{aligned} \mu_r^{"} &= \sum_{v=0}^{n} v^r P(v) \\ &= 0^r p(0) + 1^r p(1) + 2^r p(2) + \cdots + n^r p(n) \\ &= (1^r + 2^r + 3^r + \cdots + n^r)p(\{a_1\}) + (2^r + 3^r + 4^r + \cdots + n^r)p(\{a_2\}) \\ &+ \cdots + ((n-1)^r + n^r)p(\{a_{n-1}\}) + n^r p(\{a_n\}) \end{aligned} \tag{19}$$

4.2 Raw Moments based on Belief of set

As belief function is lower limit of probability, we replace probability of set by belief function of set. Therefore we get lower limit of r^{th} ordered raw moments based on probability of set as:

$$\begin{aligned}
\underline{\mu}_r^{"} &= \sum_{v=0}^{n} V^r Bel(v) \\
&= \sum_{v=1}^{n} V^r Bel(v) \quad \text{Since } Bel(0) = Bel(\emptyset) = 0 \\
&= \frac{1}{K}[(1^r + 2^r + 3^r + \cdots + n^r)p(A_1) + (2^r + 3^r + \cdots + n^r)p(A_2) \\
&+ \cdots + ((n-1)^r + n^r)p(A_{n-1}) + n^r p(A_n)] \\
&= \frac{1}{K}\sum_{u=0}^{n-1}\{\sum_{i=u}^{n-1}\{\sum_{j=0}^{n} j^r - \sum_{j=0}^{i} j^r\}\}p(A_{i+1})
\end{aligned} \tag{20}$$

4.3 Raw Moment based on Plausibility of set

As Plausibility function is upper limit of probability, we replace probability of set by belief function of set. Therefore we get upper limit of r^{th} ordered raw moments based on probability of set by (2) as:

$$\begin{aligned}
\overline{\mu}_r^{"} &= \sum_{v=0}^{n} V^r Pl(v) \\
&= \sum_{v=1}^{n} V^r Pl(v) \quad \text{Since } Pl(0) = Pl(\emptyset) = 0 \\
&= \sum_{v=1}^{n} V^r
\end{aligned} \tag{21}$$

4.4 Proper Magnification or Reduction of Upper and Lower Limits of Raw Moments

We have obtained lower and upper limits of raw moments based on probability of set. These moments are magnified or reduced by dividing corresponding raw moments based on probability of set and multiplying corresponding raw moments from concerned probability distribution. i.e.

$$\begin{gathered}
\frac{\underline{\mu}_r^{"} \cdot \mu_r^{'}}{\mu_r^{"}} \leq \mu_r^{'} \leq \frac{\overline{\mu}_r^{"} \cdot \mu_r^{'}}{\mu_r^{"}} \\
\underline{\mu}_r^{'} \leq \mu_r^{'} \leq \overline{\mu}_r^{'}
\end{gathered} \tag{22}$$

where $\mu_r^{'}$ = corresponding r^{th} raw moment of concerned probability distribution [1, 2].

Central Moments

In [1,2], we have central moments of probability distribution as:

$$\begin{aligned}
\mu_1 &= 0 \\
\mu_2 &= \mu_2^{'} - (\mu_1^{'})^2 \\
\mu_3 &= \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2(\mu_1^{'})^3 \\
\mu_4 &= \mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} - 3(\mu_1^{'})^4
\end{aligned} \tag{23}$$

Using interval arithmetic [6], raw moments (16) and corresponding lower and upper raw moments based on belief of set (17) and plausibility of set (18) respectively, we obtain lower and upper limits of central moments by replacing intervals consisting of corresponding lower or upper limits of raw moments of probability distribution as:

Lower and Upper Limits of Central Moments :-

For first central moment, we have

$$\begin{aligned}
\mu_1 &= 0 \\
&= \mu_1^{'} - \mu_1^{'} \\
&= [\underline{\mu}_1^{'}, \overline{\mu}_1^{'}] - [\underline{\mu}_1^{'}, \overline{\mu}_1^{'}] \\
&= [\underline{\mu}_1^{'} - \overline{\mu}_1^{'}, \overline{\mu}_1^{'} - \underline{\mu}_1^{'}].
\end{aligned} \tag{24}$$

For second central moment, we have

$$\begin{aligned}
\mu_2 &= \mu_2^{'} - \mu_1^{'2} \\
&= [\underline{\mu}_2^{'}, \overline{\mu}_2^{'}] - [\underline{\mu}_1^{'}, \overline{\mu}_1^{'}]^2 \\
&= [\underline{\mu}_2^{'} - \overline{\mu}_1^{'2}, \overline{\mu}_2^{'} - \underline{\mu}_1^{'2}].
\end{aligned} \tag{25}$$

For third central moment, we have

$$\begin{aligned}
\mu_3 &= \mu_3^{'} - 3(\mu_2^{'})(\mu_1^{'}) + 2\mu_1^{'2} \\
&= [\underline{\mu}_3^{'}, \overline{\mu}_3^{'}] - 3[\underline{\mu}_2^{'}, \overline{\mu}_2^{'}][\underline{\mu}_1^{'}, \overline{\mu}_1^{'}] + 2[\underline{\mu}_1^{'}, \overline{\mu}_1^{'}]^3 \\
&= [\underline{\mu}_3^{'} - 3(\overline{\mu}_2^{'})(\overline{\mu}_1^{'}) + 2\underline{\mu}_1^{'3}, \overline{\mu}_3^{'} - 3(\underline{\mu}_2^{'})(\underline{\mu}_1^{'}) + 2\overline{\mu}_1^{'3}].
\end{aligned} \tag{26}$$

For fourth central moment, we have

$$\begin{aligned}\mu_4 &= \mu_4^{'} - 4(\mu_3^{'})(\mu_1^{'}) + 6(\mu_2^{'})(\mu_1^{'})^2 - 3\mu_1^{'4}\\ &= [\underline{\mu}_4^{'}, \overline{\mu}_4^{'}] - 4[\underline{\mu}_3^{'}, \overline{\mu}_3^{'}][\underline{\mu}_1^{'}, \overline{\mu}_1^{'}]\\ &+ 6[\underline{\mu}_2^{'}, \overline{\mu}_2^{'}][\underline{\mu}_1^{'}, \overline{\mu}_1^{'}]^2 - 3[\underline{\mu}_1^{'}, \overline{\mu}_1^{'}]^4\\ &= [\underline{\mu}_4^{'} - 4\overline{\mu}_3^{'}\overline{\mu}_1^{'} + 6\underline{\mu}_2^{'}\underline{\mu}_1^{'2} - 3\overline{\mu}_1^{'4},\\ &\overline{\mu}_4^{'} - 4\underline{\mu}_3^{'}\underline{\mu}_1^{'} + 6\overline{\mu}_2^{'}\overline{\mu}_1^{'2} - 3\underline{\mu}_1^{'4}].\end{aligned} \tag{27}$$

4.5 Coefficients of Skewness and Kurtosis

Using interval arithmetic [6] and lower and upper limits of central moments (21), (22), (23) and (24), we obtain lower and upper limits of coefficient of skewness and kurtosis as:

$$\begin{aligned}\text{The Coefficient of Skewness} = \beta_1 &= \frac{\mu_3{}^2}{\mu_2{}^3}\\ &= \frac{[\underline{\mu}_3, \overline{\mu}_3]^2}{[\underline{\mu}_2, \overline{\mu}_2]^3}\\ &= \frac{[0, \overline{\mu}_3{}^2]}{[\underline{\mu}_2{}^3, \overline{\mu}_2{}^3]}\\ \text{The Coefficient of Kurtosis} = \beta_2 &= \frac{\mu_4}{\mu_2{}^2}\\ &= \frac{[\underline{\mu}_4, \overline{\mu}_4]}{[\underline{\mu}_2, \overline{\mu}_2]^2}\\ &= \frac{[\underline{\mu}_4, \overline{\mu}_4]}{[0, \overline{\mu}_2{}^2]}\end{aligned} \tag{28}$$

Remark :- Here we have obtained the lower and upper limits of central moments and hence lower and upper limits of coefficients of skewness and kurtosis. We observe that values of central moments and coefficients of skewness and kurtosis from probability distribution always lie between lower and upper limits of central moments and lower and upper limits of coefficients of skewness and kurtosis. For the value of coefficient of kurtosis $\gamma_2 = \beta_2 - 3$, we reduce the quantity β_2 by quantity 3. But in this generalization i.e. replacing a real number by suitable interval, we do not have any idea about quantity that should be replaced by quatity 3. Therefore we are unable to conclude about coefficient of kurtosis with values β_2 and γ_2.

5. Illustrative Example

Let $X \sim Binomial(n,p)$. therefore $p(x) = \binom{n}{p} p^x q^{n-x}$. Now we connsider $n = 4, p = 2/3$ and $q = 1 - p = 1/3$. The distribution of X is

X	:	0	1	2	3	4	$Total$
$p(x)$	:	1/81	8/81	24/81	32/81	16/81	1

For consonant *bba*, focal elements have embedding as :

$$\emptyset \subseteq \{0\} \subseteq \{0,1\} \subseteq \{0,1,2\} \subseteq \{0,1,2,3\} \subseteq \{0,1,2,3,4\} \tag{29}$$

Now we will calculate probability, basic belief assignment, belief, commonality and plausibility of subsets of X and represent in following table.

Subset	*Prob(·)*	*m(·)*	*Bel(·)*	*q(·)*	*Pl(·)*
$A_0 = \emptyset$	0	0	0	1=189/189	0
$A_1 = \{0\}$	1/81	1/189	1/189	1=189/189	1
$A_2 = \{0,1\}$	9/81	9/189	10/189	188/189	1
$A_3 = \{0,1,2\}$	33/81	33/189	43/189	179/189	1
$A_4 = \{0,1,2,3\}$	65/81	65/189	108/189	146/189	1
$A_5 = \{0,1,2,3,4\}$	81/81=1	81/189	189/189=1	81/189	1
$\sum$	189/81	1	351/189	783/189	5

From above table, we get lower and upper limits of distribution function of given probability distribution as belief and plausibility functions respectively, as $Bel(A_v) \leq p(A_v) \leq Pl(v), \quad v = 0,1,2,3,4,5$ including the case of subset $\emptyset$.

Now we use notation $p(v) = p(A_v) = p(\{0,1,2,3,\ldots,v-1\}) \quad v = 0,1,2,3,4,5.$

Raw Moments based on Probability, belief and plausibility of set

The r^{th} ordered raw moment based on probability of set is denoted by $\mu_r^{''}$ and is given by $\mu_r^{''} = \sum_{v=0}^{5} v^r p(v)$. Now we calculate first four raw moments based on probability of set by using values from above table $p(\cdot)$ viz. $\mu_1^{''}, \mu_2^{''}, \mu_3^{''}$ and $\mu_4^{''}$.

$$\begin{aligned} \mu_1^{"} &= \sum_{v=0}^{5} vp(v) \\ &= 0p(0) + 1p(1) + 2p(2) + 3p(3) + 4p(4) + 5p(5) = 783/81. \end{aligned} \tag{30}$$

$$\begin{aligned} \mu_2^{"} &= \sum_{v=0}^{5} v^2 p(v) \\ &= 0^2p(0) + 1^2p(1) + 2^2p(2) + 3^2p(3) + 4^2p(4) + 5^2p(5) = 3381/81. \end{aligned} \tag{31}$$

$$\begin{aligned}\mu_3^{"} &= \sum_{v=0}^{5} v^3 p(v) \\ &= 0^3p(0) + 1^3p(1) + 2^3p(2) + 3^3p(3) + 4^3p(4) + 5^3p(5) = 15249/81.\end{aligned} \tag{32}$$

$$\begin{aligned}\mu_4^{"} &= \sum_{v=0}^{5} v^4 p(v) \\ &= 0^4p(0) + 1^4p(1) + 2^4p(2) + 3^4p(3) + 4^4p(4) + 5^4p(5) \\ &= 0 \cdot 0 + 1(1/81) + 16(9/81) + 81(33/81) + 256(65/81) + 625(81/81) \\ &= 70083/81.\end{aligned} \tag{33}$$

Raw Moments based on Belief of set :-

The r^{th} ordered raw moment based on belief of set is denoted by $\underline{\mu}_r^{''}$ and is given by $\underline{\mu}_r^{''} = \sum_{v=0}^{5} v^r Bel(v)$. Now we calculate first four raw moments based on belief of set by using values from above table $Bel(\cdot)$ viz. $\underline{\mu}_1^{''}, \underline{\mu}_2^{''}, \underline{\mu}_3^{''}$ and $\underline{\mu}_4^{''}$.

$$\begin{aligned}\underline{\mu}_1^{"} &= \sum_{v=0}^{5} v Bel(v) \\ &= 0Bel(0) + 1Bel(1) + 2Bel(2) + 3Bel(3) + 4Bel(4) + 5Bel(5) = 1527/189.\end{aligned} \tag{34}$$

$$\begin{aligned}\underline{\mu}_2^{"} &= \sum_{v=0}^{5} v^2 Bel(v) \\ &= 0^2Bel(0) + 1^2Bel(1) + 2^2Bel(2) + 3^2Bel(3) + 4^2Bel(4) + 5^2Bel(5) = 6881/189.\end{aligned} \tag{35}$$

$$\begin{aligned}\underline{\mu}_3^{"} &= \sum_{v=0}^{5} v^3 Bel(v) \\ &= 0^3Bel(0) + 1^3Bel(1) + 2^3Bel(2) + 3^3Bel(3) + 4^3Bel(4) + 5^3Bel(5) = 31779/189.\end{aligned} \tag{36}$$

$$\begin{aligned}\underline{\mu}_4^{"} &= \sum_{v=0}^{5} v^4 Bel(v) \\ &= 0^4Bel(0) + 1^4Bel(1) + 2^4Bel(2) + 3^4Bel(3) + 4^4Bel(4) + 5^4Bel(5) = 149417/189.\end{aligned} \tag{37}$$

Raw Moments based on Plausibility of set :-

The r^{th} ordered raw moment based on plausibility of set is denoted by $\overline{\mu}_r^{''}$ and is given by $\overline{\mu}_r^{''} = \sum_{v=0}^{5} v^r Pl(v)$. Now we calculate first four raw moments based on plausibility of set by using values from above table $Pl(\cdot)$ viz. $\overline{\mu}_1^{''}, \overline{\mu}_2^{''}, \overline{\mu}_3^{''}$ and $\overline{\mu}_4^{''}$.

$$\begin{aligned} \overline{\mu}_1^{''} &= \sum_{v=0}^{5} vPl(v) \\ &= 0Pl(0) + 1Pl(1) + 2Pl(2) + 3Pl(3) + 4Pl(4) + 5Pl(5) = 15. \end{aligned} \tag{38}$$

$$\begin{aligned} \overline{\mu}_2^{''} &= \sum_{v=0}^{5} v^2Pl(v) \\ &= 0^2Pl(0) + 1^2Pl(1) + 2^2Pl(2) + 3^2Pl(3) + 4^2Pl(4) + 5^2Pl(5) = 55. \end{aligned} \tag{39}$$

$$\begin{aligned} \overline{\mu}_3^{''} &= \sum_{v=0}^{5} v^3Pl(v) \\ &= 0^3Pl(0) + 1^3Pl(1) + 2^3Pl(2) + 3^3Pl(3) + 4^3Pl(4) + 5^3Pl(5) = 225. \end{aligned} \tag{40}$$

$$\begin{aligned} \overline{\mu}_4^{''} &= \sum_{v=0}^{5} v^4Pl(v) \\ &= 0^4Pl(0) + 1^4Pl(1) + 2^4Pl(2) + 3^4Pl(3) + 4^4Pl(4) + 5^4Pl(5) = 979. \end{aligned} \tag{41}$$

Lower and Upper limits of Raw Moments :-

By using (31)-(38) and interval arithmetic (7)-(10), we obtain lower and upper limits of raw moments,

$$\begin{aligned} \frac{\underline{\mu}_1^{''}}{\mu_1^{''}}\mu_1^{'} \leq \mu_1^{'} &\leq \frac{\overline{\mu}_1^{''}}{\mu_1^{''}}\mu_1^{'} \\ 2.2288 \leq 2.6667 &\leq 4.1379 \\ \underline{\mu}_1^{'} \leq \mu_1^{'} &\leq \overline{\mu}_1^{'}. \end{aligned} \tag{42}$$

$$\begin{aligned} \frac{\underline{\mu}_2^{''}}{\mu_2^{''}}\mu_2^{'} \leq \mu_2^{'} &\leq \frac{\overline{\mu}_2^{''}}{\mu_2^{''}}\mu_2^{'} \\ \underline{\mu}_2^{'} \leq \mu_2^{'} &\leq \overline{\mu}_2^{'}. \end{aligned} \tag{43}$$

$$\begin{aligned} \frac{\underline{\mu}_3^{''}}{\mu_3^{''}}\mu_3^{'} \le \mu_3^{'} \le \frac{\overline{\mu}_3^{''}}{\mu_3^{''}}\mu_3^{'} \\ 23.0233 \le 25.7778 \le 30.8086 \\ \underline{\mu}_3^{'} \le \mu_3^{'} \le \overline{\mu}_3^{'}. \end{aligned} \tag{44}$$

$$\begin{aligned} \frac{\underline{\mu}_4^{''}}{\mu_4^{''}}\mu_4^{'} \le \mu_4^{'} \le \frac{\overline{\mu}_4^{''}}{\mu_4^{''}}\mu_4^{'} \\ 79.8654 \le 87.4074 \le 98.9016 \\ \underline{\mu}_4^{'} \le \mu_4^{'} \le \overline{\mu}_4^{'}. \end{aligned} \tag{45}$$

5.6 Central moments of probability Distribution

By using formulae to calculate central moments of distribution [1, 2], we have

$$\begin{aligned} &\mu_1 = 0. \\ &\mu_2 = \mu_2^{'} - {\mu_1^{'}}^2 = 0.8889. \end{aligned} \tag{46}$$

$$\mu_3 = \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2{\mu_1^{'}}^3 = 0.2963. \tag{47}$$

$$\begin{aligned} \mu_4 &= \mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}{\mu_1^{'}}^2 - 3{\mu_1^{'}}^4 \\ &= 2.0741 \end{aligned} \tag{48}$$

Lower and Upper Limits of Central Moments :- By using lower and upper limits of raw moments (39)-(42) and interval arithmetic (7)-(10), we obtain lower and upper limits of central moments as

$$\begin{aligned} \mu_1 &= [\underline{\mu}_1^{'} - \overline{\mu}_1^{'}, \overline{\mu}_1^{'} - \underline{\mu}_1^{'}] \\ &= [-1.9091, 1.9091] \\ &= [\underline{\mu}_1, \overline{\mu}_1]. \end{aligned} \tag{49}$$

$$\begin{aligned} \mu_2 &= [\underline{\mu}_2^{'} - {\overline{\mu}_1^{'}}^2, \overline{\mu}_2^{'} - {\underline{\mu}_1^{'}}^2] \\ &= [-10.1813, 5.5779] \\ &= [\underline{\mu}_2, \overline{\mu}_2]. \end{aligned} \tag{50}$$

$$\begin{aligned}\mu_3 &= [\underline{\mu}_3^{'} - 3(\overline{\mu}_2^{'})(\overline{\mu}_1^{'}) + 2\underline{\mu}_1^{'3}, \overline{\mu}_3^{'} - 3(\overline{\mu}_2^{'})(\underline{\mu}_1^{'}) + 2\overline{\mu}_1^{'3}] \\ &= [-84.9960, 128.0991] \\ &= [\underline{\mu}_3, \overline{\mu}_3].\end{aligned} \tag{51}$$

$$\begin{aligned}\mu_4 &= [\underline{\mu}_4^{'} - 4\overline{\mu}_3^{'}\overline{\mu}_1^{'} + 6\underline{\mu}_2^{'}\underline{\mu}_1^{'2} - 3\overline{\mu}_1^{'4}, \\ &\quad \overline{\mu}_4^{'} - 4\underline{\mu}_3^{'}\underline{\mu}_1^{'} + 6\overline{\mu}_2^{'}\overline{\mu}_1^{'2} - 3\underline{\mu}_1^{'4}] \\ &= [-1102.7015, 896.8144] \\ &= [\underline{\mu}_4, \overline{\mu}_4].\end{aligned} \tag{52}$$

Lower and upper Limits of Coefficients of Skewness and Kurtosis

By using lower and upper limits of central moments (46)-(49) and interval arithmetic (7)-(10), we have coefficient of skewness and kurtosis as,

$$\text{Coefficient of Skewness } = \beta_1 = \frac{\mu_3^2}{\mu_2^3} = -0.124998. \tag{53}$$

Therefore given distribution is negatively skewed as value of μ_3 is negative and it‘s sign is given to β_1 .

$$\text{Coefficient of Kurtosis } = \beta_2 = \frac{\mu_4}{\mu_2^2} = 2.624967. \tag{54}$$

Since value of coefficient of kurtosis β_2 is less than 3, the given distribution is platykurtic.

$$\begin{aligned}\text{Coefficient of Skewness} = \beta_1 &= \frac{[0, \overline{\mu}_3{}^2]}{[\underline{\mu}_2{}^3, \overline{\mu}_2{}^3]} \\ &= [-\infty, 0/(-1055.3822)] \cup [0/(173.5450), \infty] \\ &= [-\infty, 0] \cup [0, \infty] \\ &= [-\infty, \infty] \\ &= [\underline{\beta}_1, \overline{\beta}_1].\end{aligned} \tag{55}$$

$$\begin{aligned}\text{Coefficient of Kurtosis} = \beta_2 &= \frac{[\underline{\mu}_4, \overline{\mu}_4]}{[0, \overline{\mu}_2{}^2]} \\ &= \frac{[-1102.7015, 896.8144]}{[0, (5.5779)^2]} \\ &= \frac{[-1102.7015, 896.8144]}{[0, 31.11297]} \\ &= [-\infty, 896.8144/31.11297] \\ &= [-\infty, 28.8244] \\ &= [\underline{\beta}_2, \overline{\beta}_2].\end{aligned} \tag{56}$$

Remark:-

We know that sign of coefficient of skewness is dependent of sign of μ_3. From interval for μ_3, we can not conclude about sign of μ_3. Therefore we can not conclude about skewness of probability distribution. But we can have conclusion about given probability distribution that value of coefficient of skewness definitely lie in interval for coefficient of skewness.

As we do not have any idea about the quantity which corresponds to quantity 3 (which is used in statistics for conclusion about skewness) and suitably replaces quantity 3. Similar to coefficient of skewness, value of coefficient of kurtosis of probability distribution definitely lie in interval for coefficient of kurtosis.

From above two conclusions, we say that end points of intervals obtained for coefficients skewness and kurtosis represents lower and upper limits of coefficients of skewness and kurtosis. Similarly, from intervals corresponding to raw moments and central moments, the end points of respective intervals are lower and upper limits of respective raw and central moments.

6. Conclusion

In this paper, we have defined new consonant basic belief assignment and its related functions. Here lower and upper limits of statistical quantities viz. distribution function, raw moments, central moments and coefficients of skewness and kurtosis, are too much wider i.e. extreme but do not contradict statistical conclusions. The main aim of such calculations is : instead of single value, we should always prefer interval in which this value lies.

References

[1] Bansi Lal and Sanjay Arora, Mathematical Statistics, Satya Prakashan, New Delhi, (1989).

[2] Billingsley Patrick, Probability and Measures, Wiley India Pvt. Ltd., third edition, (2008).

[3] Cuzzolin F., On the Properties of The Intersection Probability, submitted to The Annals of Mathematics and Artificial Intelligence, (February 2007).

[4] Dempster A. P., Lower and upper probabilities induced by a multi-valued mapping, The Annals of Mathematical Statistics, (1966), 325-339.

[5] Dempster A. P., Lower and upper probabilities generated by a random closed interval, The Annals of Mathematical Statistics, (1968), 957-966.

[6] Moore Ramon E., Kearfott R. Baker and Cloud Michael J., Introduction to interval analysis, Society for Industrial and Applied Mathematics, Philadephia, PA, USA, (2009).

[7] Shafer Glenn, A Mathematical Theory of Evidence, Princeton University Press, NJ, (1976).

[8] Shannon C. E., A Mathematical theory of communication, Bell Syst. Tech. J., 27 (1948), 379-423 and 623-656.

[9] Smets Philippe, Constructing the pignistic probability function in a context of uncertainty, Uncertainty in Artificial Intelligence, 5 (1990), 29-39.

[10] Sudano J., Pignistic probability transforms for mixes of low- and high-probability events, In Proceedings of Fusion 2001 International Conference on Information Fusion, Montreal, Canada, (August 2001).

[11] Sudano J., The system probability information content (PIC) relationship to contributing components, combining independent multi source beliefs, hybrid and peddigree pignistic probabilities, In Proceedings of Fusion 2002 International Conference on Information Fusion, Annapolis, Maryland, U.S.A., 2 (July 2002), 1277-1283.

[12] Sudano J., Yet Another Paradigm Illustrating Evidence Fusion (YAPIEF), In Proceedings of Fusion 2006 International Conference on Information Fusion, Italy, (July 2006).