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SOME BASIC PROPERTIES OF MULTIVALENT FUNCTIONS DEFINED BY GENERALIZED BERNARDI-LIBERA-LIVINGSTON INTEGRAL OPERATOR

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Abstract

By means of certain differential operator we introduce and investigate two subclasses $S^q_{n,m}(\lambda, b, \delta)$ and $G^q_{n,m}(\lambda, b, \delta)$ of q-valently analytic functions. The various results obtained here for each of these classes we have attempted to obtain coefficient estimate, growth and distortion theorems, for the classes $S^q_{n,m}(\lambda, b, \delta)$ and $G^q_{n,m}(\lambda, b, \delta)$.

1. Introduction

This chapter introduces p-valent functions and its various properties. By means of certain Generalized Bernardi-Libera-Livingston Integral Operator,we introduce and investigate two new subclasses $S^p_{n,m}(\lambda, b, \delta)$ and $G^p_{n,m}(\lambda, b, \delta)$ of p-valently analytic functions of complex order. The various results obtained here for each of these subclasses.

Let $A(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad a_k \geq 0 \text{ and } n, p \in \mathbb{N} = \{1, 2, 3, \cdots\} \tag{1.1}$$

Key Words and Phrases : *Multivalent function, Coefficient estimate, Distortion theorem.*

which are analytic and p-valent in the open unit disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.
We introduce here Generalized Bernardi-Libera-Livingston Integral Operator:

$$\mathcal{F}_p^\lambda f(z) = \frac{\lambda + p}{z^\lambda} \int_0^z x^{\lambda-1} f(x)dx, \quad (\lambda > -p; z \in U).$$

Simplifying we get

$$\mathcal{F}_p^\lambda f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{\lambda + p}{\lambda + k} a_k z^k,$$

$$(\mathcal{F}_p^\lambda f(z))^{(m)} = \begin{pmatrix} p \\ m \end{pmatrix} z^{p-m} - \sum_{k=n+p}^{\infty} \begin{pmatrix} k \\ m \end{pmatrix} \left(\frac{\lambda + p}{\lambda + k}\right) a_k z^{k-m}$$

where $\begin{pmatrix} k \\ m \end{pmatrix} = \frac{k(k-1)(k-2)\cdots(k-m+1)}{m!}$.
By using the operator $\mathcal{F}_p^\lambda(z)$, we introduce new subclass $S_{n,m}^p(\lambda, b, \delta)$ of p-valently analytic function $f(z)$ satisfying the following inequality

$$\left| \frac{1}{b} \left(\frac{\delta z(\mathcal{F}_p^\lambda f(z))^{(m+1)} + \lambda z^2 (\mathcal{F}_p^\lambda f(z))^{(m+2)}}{\lambda z(\mathcal{F}_p^\lambda f(z))^{(m+1)} + (\delta - \lambda)(\mathcal{F}_p^\lambda f(z))^{(m)}} - (p - m) \right) \right| < 1 \tag{1.2}$$

$p \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}, z \in E, p > \max(m - \lambda), b \in \mathbb{C}U\{0\}, \lambda \geq 0, 0 < \delta \leq 1$.
Furthermore a function $f(z)$ is said to belong to the class $G_{n,m}^p(\lambda, b, \delta)$ if and only if

$$zf'(z) \in S_{n,m}^p(\lambda, b, \delta).$$

The objective of this paper is to investigate the various properties and characteristics of analytic p-valent functions belonging to the subclasses $S_{n,m}^p(\lambda, b, \delta)$ and $G_{n,m}^p(\lambda, b, \delta)$ which we have defined here. Apart from deriving a set of coefficient bounds for each of these function classes, we establish growth and distortion theorem.
Our definitions of the function classes $S_{n,m}^p(\lambda, b, \delta)$ and $G_{n,m}^p(\lambda, b, \delta)$ are motivated by the investigation of H. M. Srivastava and others [2], we have relaxed the parametric constraint $0 \leq \lambda \leq 1$.

2. Coefficient Estimates

Theorem 2.2.1 : A function $f(z) \in A(n)$ and defined by $f(z) = z^p - \sum\limits_{k=n+p}^{\infty} a_k z^k, a_k > 0$ and $p \in \mathbb{N}$, is in $S_{n,m}^p(\lambda, b, \delta)$ if and only if

$$\sum_{k=n+p}^{\infty} \left(\frac{\lambda + p}{\lambda + k}\right) \begin{pmatrix} k \\ m \end{pmatrix} [\lambda(k - m - 1) + \delta][k - p + |b|]a_k$$

$$\leq |b| \binom{p}{m} [\lambda(p-m-1)+\delta] \tag{2.2.1}$$

Proof : Suppose that $f(z) \in S^p_{n,m}(\lambda, b, \delta)$. Therefore we have

$$\left| \frac{1}{b} \left(\frac{\delta z(\mathcal{F}^\lambda_p f(z))^{(m+1)} + \lambda z^2 (\mathcal{F}^\lambda_p f(z))^{(m+2)}}{\lambda z(\mathcal{F}^\lambda_p f(z))^{(m+1)} + (\delta-\lambda)(\mathcal{F}^\lambda_p f(z))^{(m)}} - (p-m) \right) \right| < 1 \tag{2.2.2}$$

$$(\mathcal{F}^\lambda_p f(z))^{(m)} = \binom{p}{m} z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m} \left(\frac{\lambda+p}{\lambda+k}\right) a_k z^{k-m}$$

$$(\mathcal{F}^\lambda_p f(z))^{(m+1)} = \binom{p}{m} (p-m) z^{p-m-1} - \sum_{k=n+p}^{\infty} \binom{k}{m} \left(\frac{\lambda+p}{\lambda+k}\right) (k-m) a_k z^{k-m-1}$$

$$(\mathcal{F}^\lambda_p f(z))^{(m+2)} = \binom{p}{m} (p-m)(p-m-1) z^{p-m-2}$$
$$- \sum_{k=n+p}^{\infty} \binom{k}{m} \left(\frac{\lambda+p}{\lambda+k}\right) (k-m)(k-m-1) a_k z^{k-m-2}$$

$$\delta z(\mathcal{F}^\lambda_p f(z))^{(m+1)} + \lambda z^2 (\mathcal{F}^\lambda_p f(z))^{(m+2)}$$
$$= \binom{p}{m} (p-m)[\delta + \lambda(p-m-1) z^{p-m}$$
$$- \sum_{k=n+p}^{\infty} \binom{k}{m} \left(\frac{\lambda+p}{\lambda+k}\right) (k-m)[\delta+\lambda(k-m-1)] a_k z^{k-m}$$

$$\lambda z(\mathcal{F}^\lambda_p f(z))^{(m+1)} + (\delta-\lambda)(\mathcal{F}^\lambda_p f(z))^{(m)}$$
$$= \binom{p}{m} [\lambda(p-m)+\delta-\lambda] z^{p-m}$$
$$- \sum_{k=n+p}^{\infty} \binom{k}{m} \left(\frac{\lambda+p}{\lambda+k}\right) [\lambda(k-m)+\delta-\lambda] a_k z^{k-m}.$$

From (2.2.2) we have

$$\left| \frac{1}{b} \left(\frac{\sum_{k=n+p}^{\infty} \binom{k}{m} \left(\frac{\lambda+p}{\lambda+k}\right) [\lambda(k-m-1)+\delta][-k+m+p-m] a_k z^{k-m}}{\binom{p}{m} [\lambda(p-m-1)+\delta] z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m} \left(\frac{\lambda+p}{\lambda+k}\right) [\lambda(k-m-1)+\delta] a_k z^{k-m}} \right) \right| < 1. \tag{2.2.3}$$

We know that $Re(z) < |z|$, also putting $z = r_1, 0 \leq r_1 \leq 1$ in (1.3), we observe that expression in the denominator on left hand side of (1.3) is positive for $r_1 = 0$ and by letting $r_1 \to 1_-$ through real values, (2.2.3) leads us that

$$\frac{1}{|b|}\left[\frac{\sum_{k=n+p}^{\infty}\binom{k}{m}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta][k-p]a_k}{\binom{p}{m}[\lambda(p-m-1)+\delta]-\sum_{k=n+p}^{\infty}\binom{k}{m}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta]a_k}\right] < 1.$$

$$\sum_{k=n+p}^{\infty}\binom{k}{m}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta][k-p+|b|]a_k \leq |b|\binom{p}{m}[\lambda(p-m-1)+\delta].$$

Hence we get (2.2.1) Conversely, by applying (2.1.2) and setting $|z| = 1$ we find that

$$\left|\left(\frac{\delta z(\mathcal{F}_p^{\lambda}f(z))^{(m+1)}+\lambda z^2(\mathcal{F}_p^{\lambda}f(z))^{(m+2)}}{\lambda z(\mathcal{F}_p^{\lambda}f(z))^{(m+1)}+(\delta-\lambda)(\mathcal{F}_p^{\lambda}f(z))^{(m)}}-(p-m)\right)\right|$$

$$=\left|\left(\frac{\sum_{k=n+p}^{\infty}\binom{k}{m}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta][k-p]a_kz^{k-m}}{\binom{p}{m}[\lambda(p-m-1)+\delta]-\sum_{k=n+p}^{\infty}\binom{k}{m}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta]a_kz^{k-m}}\right)\right|$$

$$\leq |b|\frac{\binom{p}{m}[\lambda(p-m-1)+\delta]z^{p-m}-\sum_{k=n+p}^{\infty}\binom{k}{m}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta]a_kz^{k-m}}{\binom{p}{m}[\lambda(p-m-1)+\delta]z^{p-m}-\sum_{k=n+p}^{\infty}\binom{k}{m}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta]a_kz^{k-m}} = |b|.$$

Hence by the maximum modulus principle, we infer that $f(z) \in S^p_{n,m}(\lambda, b, \delta)$ which completes the proof of Theorem 2.2.1.

Corollary 2.2.1 : $f(z) \in S^p_{n,m}(\lambda, b, \delta)$ then

$$a_k \leq \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{k}{m}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta][k-p+|b|]}.$$

Corollary 2.2.2 : for $p = 1$, $m = 0$ we have

$$a_k \leq \frac{|b|\delta(\lambda+k)}{(\lambda+1)[\lambda(k-m-1)+\delta][k-1+|b|]}, k \geq n+p.$$

Corollary 2.2.3 : For $p = 1$, $m = 1$ we have

$$a_k \leq \frac{|b|[\delta-\lambda](\lambda+k)}{k(\lambda+1)[\lambda+\delta][k-1+|b|]}.$$

Theorem 2.2.2 : A function $f(z) \in A(n)$ and defined by $f(z) = z^p - \sum_{k=n+p}^{\infty} ka_k z^k and$ p $\in \mathbb{N}$, is in $G^p_(n,m)(\lambda,b,\delta)$ if and only if

$$\sum_{k=n+p}^{\infty} \left(\frac{\lambda+p}{\lambda+k}\right) \left(\begin{array}{c} k \\ m \end{array} \right) k[\lambda - m-1)+\delta][k-p+|b|] \leq |b| \left(\begin{array}{c} p \\ m \end{array} \right) p[\lambda(p-m-1)+\delta]. \quad (2.2.4)$$

Proof : $f(z) \in G^p_{n,m}(\lambda,b,\delta)$ if and only if $zf'(z) \in S^p_{n,m}(\lambda,b,\delta)$. Let

$$g(z) = zf^{'(z)} = pz^p - \sum_{k=n+p}^{\infty} ka_k z^k$$

$$g(z) \in S^p_{n,m}(\lambda,b,\delta).$$

Therefore

$$\left| \frac{1}{b} \left(\frac{\delta z(\mathcal{F}^\lambda_p(g(z)))^{(m+1)} + \lambda z^2 (\mathcal{F}^\lambda_p(g(z)))^{(m+2)}}{\lambda z(\mathcal{F}^\lambda_p(g(z)))^{(m+1)} + (\delta-\lambda)(\mathcal{F}^\lambda_p(g(z)))^{(m)}} - (p-m) \right) \right| < 1 \qquad (2.2.5)$$

$$(\mathcal{F}^\lambda_p(g(z)))^{(m)} = \left(\begin{array}{c} p \\ m \end{array} \right) pz^{p-m} - \sum_{k=n+p}^{\infty} \left(\begin{array}{c} k \\ m \end{array} \right) \left(\frac{\lambda+p}{\lambda+k}\right) ka_k z^{k-m}$$

$$(\mathcal{F}^\lambda_p(g(z)))^{(m+1)} = \left(\begin{array}{c} p \\ m \end{array} \right) p(p-m)z^{p-m-1} - \sum_{k=n+p}^{\infty} \left(\begin{array}{c} k \\ m \end{array} \right) \left(\frac{\lambda+p}{\lambda+k}\right) (k-m)ka_k z^{k-m-1}$$

$$(\mathcal{F}^\lambda_p g(z))^{(m+2)} = \left(\begin{array}{c} p \\ m \end{array} \right) p(p-m)(p-m-1)z^{p-m-2}$$
$$- \sum_{k=n+p}^{\infty} \left(\begin{array}{c} k \\ m \end{array} \right) \left(\frac{\lambda+p}{\lambda+k}\right) (k-m)(k-m-1)ka_k z^{k-m-2}$$

$$\delta z(\mathcal{F}^\lambda_p g(z))^{(m+1)} + \lambda z^2 (\mathcal{F}^\lambda_p(g(z)))^{(m+2)}$$
$$= \left(\begin{array}{c} p \\ m \end{array} \right) p(p-m)[\delta + \lambda(p-m-1)z^{p-m}$$
$$- \sum_{k=n+p}^{\infty} \left(\begin{array}{c} k \\ m \end{array} \right) \left(\frac{\lambda+p}{\lambda+k}\right) k(k-m)[\delta+\lambda(k-m-1)]a_k z^{k-m}$$

Now consider

$$\lambda z(\mathcal{F}^\lambda_p(g(z))^{(m+1)} + (\delta-\lambda)(\mathcal{F}^\lambda_p(g(z))^{(m)}$$
$$= p \left(\begin{array}{c} p \\ m \end{array} \right) [\lambda(p-m)+\delta-\lambda]z^{p-m}$$
$$- \sum_{k=n+p}^{\infty} \left(\begin{array}{c} k \\ m \end{array} \right) \left(\frac{\lambda+p}{\lambda+k}\right) k[\lambda(k-m)+\delta-\lambda]a_k z^{k-m}.$$

From (2.2.5) we have

$$\left|\frac{1}{b}\left(\frac{\sum_{k=n+p}^{\infty}\begin{pmatrix}k\\m\end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta][-k+p]a_k z^{k-m}}{\begin{pmatrix}p\\m\end{pmatrix}p[\lambda(p-m-1)+\delta]z^{p-m}-\sum_{k=n+p}^{\infty}\begin{pmatrix}k\\m\end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)k[\lambda(k-m-1)+\delta]a_k z^{k-m}}\right)\right|<1. \tag{2.2.6}$$

We know that $Re(z) < |z|$, also putting $z = r_1, 0 \le r_1 \le 1$ in (2.2.6), we observe that expression in the denominator on left hand side of (2.3) is positive for $r_1 = 0$ and by letting $r_1 \to 1_-$ through real values, (2.2.6) leads us that

$$\frac{1}{|b|}\left[\frac{\sum_{k=n+p}^{\infty}\begin{pmatrix}k\\m\end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta]k[k-p]a_k}{p\begin{pmatrix}p\\m\end{pmatrix}[\lambda(p-m-1)+\delta]-\sum_{k=n+p}^{\infty}\begin{pmatrix}k\\m\end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)k[\lambda(k-m-1)+\delta]a_k}\right]\le 1.$$

$$\sum_{k=n+p}^{\infty}\begin{pmatrix}k\\m\end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)k[\lambda(k-m-1)+\delta][k-p+|b|]a_k \le |b|p\begin{pmatrix}p\\m\end{pmatrix}[\lambda(p-m-1)+\delta].$$

Hence we get (2.2.4).

Conversely, by applying (2.2.4) and setting $|z| = 1$ we find that

$$\left|\left(\frac{\delta z(\mathcal{F}_p^{\lambda}(g(z)))^{(m+1)}+\lambda z^2(\mathcal{F}_p^{\lambda}(g(z)))^{(m+2)}}{\lambda z(\mathcal{F}_p^{\lambda}(g(z)))^{(m+1)}+(\delta-\lambda)(\mathcal{F}_p^{\lambda}(g(z)))^{(m)}}-(p-m)\right)\right|$$

$$=\left|\left(\frac{\sum_{k=n+p}^{\infty}\begin{pmatrix}k\\m\end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)k[\lambda(k-m-1)+\delta][k-p]a_k z^{k-m}}{p\begin{pmatrix}p\\m\end{pmatrix}[\lambda(p-m-1)+\delta]-\sum_{k=n+p}^{\infty}\begin{pmatrix}k\\m\end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)k[\lambda(k-m-1)+\delta]a_k z^{k-m}}\right)\right|$$

$$\le |b|\frac{p\begin{pmatrix}p\\m\end{pmatrix}[\lambda(p-m-1)+\delta]z^{p-m}-\sum_{k=n+p}^{\infty}\begin{pmatrix}k\\m\end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)k[\lambda(k-m-1)+\delta]a_k z^{k-m}}{p\begin{pmatrix}p\\m\end{pmatrix}[\lambda(p-m-1)+\delta]z^{p-m}-\sum_{k=n+p}^{\infty}\begin{pmatrix}k\\m\end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)k[\lambda(k-m-1)+\delta]a_k z^{k-m}}=|b|.$$

Hence by the maximum modulus principle, we infer that $g(z) \in S_{n,m}^{p}(\lambda, b, \delta)$ which completes the proof of Theorem 2.2.2.

Corollary 2.2.4 : For $f(z) \in G_{n,m}^{p}(\lambda, b, \delta)$ then

$$a_k \le \frac{|b|p\begin{pmatrix}p\\m\end{pmatrix}[\lambda(p-m-1)+\delta]}{\begin{pmatrix}k\\m\end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)k[\lambda(k-m-1)+\delta][k-p+|b|]}.$$

2.3 Growth and Distortiion Theorem

Theorem 2.3.1 : If $f(z) \in S^p_{n,m}(\lambda, b, \delta)$ then

$$|z|^p - |z|^{n+p} \frac{|b| \begin{pmatrix} p \\ m \end{pmatrix} [\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+k}\right) [\lambda(n+p-m-1)+\delta][n+|b|]} \leq |f(z)|$$

$$\leq |z|^p + |z|^{n+p} \frac{|b| \begin{pmatrix} p \\ m \end{pmatrix} [\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+k}\right) [\lambda(n+p-m-1)+\delta][n+|b|]}.$$

With equality hold for

$$f(z) = z^p - z^{n+p} \frac{|b| \begin{pmatrix} p \\ m \end{pmatrix} [\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+k}\right) [\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Proof : $f(z) \in S^p_{n,m}(\lambda, b, \delta)$. Therefore from (2.2.1)

$$\sum_{k=n+p}^{\infty} \left(\frac{\lambda+p}{\lambda+k}\right) \begin{pmatrix} k \\ m \end{pmatrix} [\lambda(k-m-1)+\delta][k-p+|b|] \leq |b|p \begin{pmatrix} p \\ m \end{pmatrix} [\lambda(p-m-1)+\delta].$$

Therefore

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b| \begin{pmatrix} p \\ m \end{pmatrix} [\lambda(p-m-1)+\delta]}{\begin{pmatrix} k \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+k}\right) [\lambda(k-m-1)+\delta][k-p+|b|]}$$

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$|f(z)| \geq |z|^p - \sum_{k=n+p}^{\infty} |a_k||z|^k \geq |z|^p - |z|^{n+p} \sum_{k=n+p}^{\infty} |a_k|$$

$$\geq |z|^p - |z|^{n+p} \frac{|b| \begin{pmatrix} p \\ m \end{pmatrix} [\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+n+p}\right) [\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Similarly

$$|f(z)| \leq |z|^p + \sum_{k=n+p}^{\infty} |a_k||z|^k \leq |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} |a_k|$$
$$\leq |z|^p + |z|^{n+p} \frac{|b| \binom{p}{m} [\lambda(p-m-1)+\delta]}{\binom{n+p}{m} \left(\frac{\lambda+p}{\lambda+n+p}\right) [\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Therefore

$$|z|^p - |z|^{n+p} \frac{|b| \binom{p}{m} [\lambda(p-m-1)+\delta]}{\binom{n+p}{m} \left(\frac{\lambda+p}{\lambda+k}\right) [\lambda(n+p-m-1)+\delta][n+|b|]} \leq |f(z)|$$
$$\leq |z|^p + |z|^{n+p} \frac{|b| \binom{p}{m} [\lambda(p-m-1)+\delta]}{\binom{n+p}{m} \left(\frac{\lambda+p}{\lambda+n+p}\right) [\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Theorem 2.3.2 : If $f(z) \in G^p_{n,m}(\lambda, b, \delta)$ then

$$|z|^p - |z|^{n+p} \frac{|b|p \binom{p}{m} [\lambda(p-m-1)+\delta]}{\binom{n+p}{m} \left(\frac{\lambda+p}{\lambda+n+p}\right) (n+p)[\lambda(n+p-m-1)+\delta][n+|b|]} \leq |f(z)|$$
$$\leq |z|^p + |z|^{n+p} \frac{|b|p \binom{p}{m} [\lambda(p-m-1)+\delta]}{\binom{n+p}{m} \left(\frac{\lambda+p}{\lambda+n+p}\right) (n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

With equality hold for

$$f(z) = z^p - z^{n+p} \frac{|b|p \binom{p}{m} [\lambda(p-m-1)+\delta]}{\binom{n+p}{m} \left(\frac{\lambda+p}{\lambda+n+p}\right) [\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Proof : $f(z) \in G^p_{n,m}(\lambda, b, \delta)$. Therefore from Theorem 2.2.2

$$\sum_{k=n+p}^{\infty} \left(\frac{\lambda+p}{\lambda+k}\right) \binom{k}{m} k[\lambda(k-m-1)+\delta][k-p+|b|] \leq |b|p \binom{p}{m} [\lambda(p-m-1)+\delta].$$

Therefore

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b|p\begin{pmatrix} p \\ m \end{pmatrix}[\lambda(p-m-1)+\delta]}{\begin{pmatrix} k \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)k[\lambda(k-m-1)+\delta][k-p+|b|]}$$

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$|f(z)| \geq |z|^p - \sum_{k=n+p}^{\infty} |a_k||z|^k \geq |z|^p - |z|^{n+p} \sum_{k=n+p}^{\infty} |a_k|$$

$$\geq |z|^p - |z|^{n+p} \frac{|b|p\begin{pmatrix} p \\ m \end{pmatrix}[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Similarly

$$|f(z)| \leq |z|^p + \sum_{k=n+p}^{\infty} |a_k||z|^k \leq |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} |a_k|$$

$$\leq |z|^p + |z|^{n+p} \frac{|b|p\begin{pmatrix} p \\ m \end{pmatrix}[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Therefore

$$|z|^p - |z|^{n+p} \frac{|b|p\begin{pmatrix} p \\ m \end{pmatrix}[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]} \leq |f(z)|$$

$$\leq |z|^p + |z|^{n+p} \frac{|b|p\begin{pmatrix} p \\ m \end{pmatrix}[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Theorem 2.3.3 : If $f(z) \in S^p_{n,m}(\lambda, b, \delta)$ then

$$p|z|^{p-1} - |z|^{n+p-1}\frac{|b|\begin{pmatrix} p \\ m \end{pmatrix}(n+p)[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)[\lambda(n+p-m-1)+\delta][n+|b|]} \leq |f'(z)|$$

$$\leq p|z|^{p-1} + |z|^{n+p-1}\frac{|b|\begin{pmatrix} p \\ m \end{pmatrix}(n+p)[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

With equality hold for

$$f(z) = z^p - z^{n+p}\frac{|b|\begin{pmatrix} p \\ m \end{pmatrix}[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Proof : $f(z) \in S^p_{n,m}(\lambda, b, \delta)$. Therefore from (2.2.1)

$$\sum_{k=n+p}^{\infty}\left(\frac{\lambda+p}{\lambda+k}\right)\begin{pmatrix} k \\ m \end{pmatrix}[\lambda(k-m-1)+\delta][k-p+|b|] \leq |b|\begin{pmatrix} p \\ m \end{pmatrix}[\lambda(p-m-1)+\delta].$$

Therefore

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b|\begin{pmatrix} p \\ m \end{pmatrix}[\lambda(p-m-1)+\delta]}{\begin{pmatrix} k \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right)[\lambda(k-m-1)+\delta][k-p+|b|]}$$

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$f'(z) = pz^{p-1} - \sum_{k=n+p}^{\infty} a_k k z^{k-1}$$

$$|f'(z)| \geq p|z|^{p-1} - \sum_{k=n+p}^{\infty}|a_k||k||z|^{k-1} \geq p|z|^{p-1} - |z|^{n+p-1}(n+p)\sum_{k=n+p}^{\infty}|a_k|$$

$$\geq p|z|^{p-1} - |z|^{n+p-1}\frac{|b|\begin{pmatrix} p \\ m \end{pmatrix}(n+p)[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Similarly

$$|f'(z)| \leq |z|^{p-1} + \sum_{k=n+p}^{\infty} |a_k| k |z|^{k-1} \leq p|z|^{p-1} + |z|^{n+p-1}(n+p) \sum_{k=n+p}^{\infty} |a_k|$$

$$\leq |z|^{p-1} + |z|^{n+p-1} \frac{|b| \begin{pmatrix} p \\ m \end{pmatrix} (n+p)[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+n+p}\right) [\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Therefore

$$p|z|^{p-1} - |z|^{n+p-1} \frac{|b| \begin{pmatrix} p \\ m \end{pmatrix} (n+p)[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+n+p}\right) [\lambda(n+p-m-1)+\delta][n+|b|]} \leq |f'(z)|$$

$$\leq p|z|^{p-1} + |z|^{n+p-1} \frac{|b| \begin{pmatrix} p \\ m \end{pmatrix} (n+p)[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+n+p}\right) [\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Theorem 2.3.4 : If $f(z) \in G^p_{n,m}(\lambda, b, \delta)$ then

$$p|z|^{p-1} - |z|^{n+p-1} \frac{|b| \begin{pmatrix} p \\ m \end{pmatrix} p[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+n+p}\right) [\lambda(n+p-m-1)+\delta][n+|b|]} \leq |f'(z)|$$

$$\leq p|z|^{p-1} + |z|^{n+p-1} \frac{|b| \begin{pmatrix} p \\ m \end{pmatrix} p[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+n+p}\right) [\lambda(n+p-m-1)+\delta][n+|b|]}.$$

With equality hold for

$$f(z) = z^p - z^{n+p} \frac{|b|p \begin{pmatrix} p \\ m \end{pmatrix} [\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix} \left(\frac{\lambda+p}{\lambda+n+p}\right) (n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Proof : $f(z) \in G^p_{n,m}(\lambda, b, \delta)$. Therefore from Theorem 2.2.2

$$\sum_{k=n+p}^{\infty} \left(\frac{\lambda+p}{\lambda+k}\right) \begin{pmatrix} k \\ m \end{pmatrix} k[\lambda(k-m-1)+\delta][k-p+|b|] \leq |b|p \begin{pmatrix} p \\ m \end{pmatrix} [\lambda(p-m-1)+\delta].$$

Therefore

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b|\begin{pmatrix} p \\ m \end{pmatrix} p[\lambda(p-m-1)+\delta]}{\begin{pmatrix} k \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+k}\right) k[\lambda(k-m-1)+\delta][k-p+|b|]}$$

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$f'(z) = pz^{p-1} - \sum_{k=n+p}^{\infty} a_k k z^{k-1}$$

$$|f'(z)| \geq p|z|^{p-1} - \sum_{k=n+p}^{\infty} |a_k| k |z|^{k-1} \geq p|z|^{p-1} - |z|^{n+p-1}(n+p) \sum_{k=n+p}^{\infty} |a_k|$$

$$\geq p|z|^{p-1} - |z|^{n+p-1} \frac{|b|\begin{pmatrix} p \\ m \end{pmatrix} p[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Similarly

$$|f'(z)| \leq |z|^{p-1} + \sum_{k=n+p}^{\infty} |a_k| k |z|^{k-1} \leq p|z|^{p-1} + |z|^{n+p-1}(n+p) \sum_{k=n+p}^{\infty} |a_k|$$

$$\leq p|z|^{p-1} + |z|^{n+p-1} \frac{|b|\begin{pmatrix} p \\ m \end{pmatrix} p[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

Therefore

$$p|z|^{p-1} - |z|^{n+p-1} \frac{|b|\begin{pmatrix} p \\ m \end{pmatrix} p[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)[\lambda(n+p-m-1)+\delta][n+|b|]} \leq |f'(z)|$$

$$\leq p|z|^{p-1} + |z|^{n+p-1} \frac{|b|\begin{pmatrix} p \\ m \end{pmatrix} p[\lambda(p-m-1)+\delta]}{\begin{pmatrix} n+p \\ m \end{pmatrix}\left(\frac{\lambda+p}{\lambda+n+p}\right)[\lambda(n+p-m-1)+\delta][n+|b|]}.$$

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