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FIXED POINT TECHNIQUE FOR SOLVING A GENERALIZED SET-VALUED IMPLICIT QUASI-VARIATIONAL INEQUALITY PROBLEM

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Abstract

In this paper, we consider a generalized set-valued implicit quasi-variational inequality problem (GSIQVIP) in real uiniformly smooth Banach space. Using set-valued version of Boyd-Wong fixed point theorem [3], we prove the existence of solution for GSIQVIP. By exploiting the method of this paper, one can generalize and improve many known results in the literature.

1. Introduction

In 1973, Benssousan *et al.* [2] introduced a new class of variational inequalities known as quasi-variational inequalities arising in the study of optimization, economics and

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impulse control theory. In the variational inequality formulation, the underlying convex set K does not depend upon the solution. In many important applications, the convex set K depends implicitly on the solution. In this case, variational inequality is known as quasi-variational inequality, see for the details [1, 2, 5, 6, 10-12].
The results concerning for solving monotone set-valued variational inequalities established by many authors are actually for single-valued variational inequalities inspite of involving set-valued mappings. Therefore, such methods used for studying the existence of solutions for set-valued variational inequalities need improvement. In 1999, Noor [11] considered a class of quasi-variational inequalities involving set-valued mappings with compact values in Hilbert space which is called the set-valued implicit quasi-variational inequalities. Using fixed point technique and projection method, he studied the existence of solution for a class of set-valued implicit quasi-variational inequalities.
Recently, many authors given in [1, 6, 8, 10-13] studied existence of solutions for some classes of variational inequalities involving single-valued and set-valued mappings in Banach spaces using some improved techniques. Therefore, it is an interesting problem to generalize and improve the techniques developed by some authors given in [1, 6, 8, 10-12] to study the set-valued implicit quasi-variational inequality problem in real uniformly smooth Banach space under some weak conditions.

Inspired by recent research work in this direction, we consider a generalized set-valued implicit quasi-variational inequality problem (GSIQVIP) in real uniformly smooth Banach space. Using set-valued version of Boyd-Wong fixed point theorem [3], we prove the existence of solution for GSIQVIP.

2. Preliminaries

Let E be a real Banach space equipped with norm $\|\cdot\|$; E^* be the topological dual space of E; $\langle\cdot,\cdot\rangle$ be the dual pair between E and E^*; $CB(E)$ be the family of all nonempty, closed and bounded subsets of E. Let $\mathcal{H}(\cdot,\cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$\mathcal{H}(A,B) \;=\; \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\right\}; \qquad A,B\in CB(E);$$

and $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) \;=\; \{f\in E^* :\ \ \langle x,\ f\rangle \;=\; \|x\|^2,\, \|x\| \;=\; \|f\|\}, \quad \forall x\in E.$$

First, we recall and define the following known concepts and results.

Definition 2.1 [4, 7, 13] : A Banach space E is called smooth if for every $x \in E$ with $\|x\| = 1$, there exists a unique $f \in E^*$ such that $\|f\| = f(x) = 1$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$, defined by

$$\rho_E(\tau) \ = \ \sup\Big\{\frac{(\|x+y\| + \|x-y\|)}{2} - 1 : \ x, y \in E, \ \|x\| = 1, \ \|y\| = \tau\Big\}.$$

Definition 2. 2 [4] : The space E is said to be uniformly smooth if, $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$.

Remark 2.1 [4] : We note that if E is smooth then the normalized duality mapping J is single-valued and if $E \equiv H$, a Hilbert space, then J becomes identity.

Lemma 2.1 [4, 7, 13] : Let E be an uniformly smooth Banach space and let $J : E \to E^*$ be the normalized duality mapping. Then for all $x, y \in E$, we have

(i) $\|x+y\|^2 \ \leq \ \|x\|^2 + 2\langle y, \ J(x+y)\rangle$;

(ii) $\langle x-y, \ J(x) - J(y)\rangle \ \leq \ 2d^2\rho_E(4\|x-y\|/d)$, where $d = \sqrt{(\|x\|^2 + \|y\|^2)/2}$.

Definition 2.3 [9-12] : A set-valued mapping $T : E \to CB(E)$ is said to be μ-$\mathcal{H}$-Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\mathcal{H}(T(x), \, T(y)) \ \leq \ \mu\|x-y\|, \quad \forall \, x, y \in E,$$

where $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(E)$.

Theorem 2.1 [9-12] :

(i) Let $T : E \to CB(E)$ be a set-valued mapping on E. Then for any given $\epsilon > 0$ and for any $x, y \in E$ and $u \in T(x)$, there exists $v \in T(y)$ such that

$$\|u - v\| \leq (1 + \epsilon)\mathcal{H}(T(x), T(y));$$

(ii) If $T : E \to C(E)$, then above inequality holds for $\epsilon = 0$.

Definition 2.4 [1,7] : Let K be a nonempty, closed and convex subset of uniformly smooth Banach space E. A mapping $R_K : E \to K$ is said to be:

(i) retraction if $R_K^2 = R_K$;

(ii) nonexpansive retraction if $R_K(x) - R_K(y)\| \leq \|x - y\|, \quad \forall x, y \in E$;

(iii) sunny retraction if $R_K(R_K(x) - t(x - R_K(x))) = R_K(x), \quad \forall x \in E,\ t \in \mathbb{R}$.

Theorem 2.2 [1,7] : Let E be a uniformly smooth Banach space and let $J : E \to E^*$ be the normalized duality mapping. Then R_K is sunny nonexpansive retraction if and only if for all $x, y \in E$, we have

$$\langle x - R_K(x), J(R_K(x) - y)\rangle \geq 0. \tag{2.1}$$

Definition 2.5 [3,9] : Let $F : X \to X$ be a mapping; X is a metric space with metric $d(\cdot,\cdot)$; $P := \{d(x,y) : x, y \in X\}$ and let $\bar{P}$ denote the closure of P. Then:

(i) A point $x \in X$ is said to be fixed point of F if $F(x) = x$;

(ii) F is said to be contraction if $d(F(x), F(y)) \leq \alpha d(x,y), \quad \forall x, y \in X,$ for some $\alpha,\ 0 \leq \alpha < 1$. Further, if $\alpha = 1$, then F is called nonexpansive.

Theorem 2.3 [3,9] : **(Banach Contraction Principle).** Every contraction mapping F defined on a complete metric space X has a unique fixed point.

Definition 2.6 [3, 13] : A metric space X is said to be metrically convex if for each $x, y \in X$ with $x \neq y$, there is a $z \in X,\ x \neq z \neq y$ such that $d(x,y) = d(x,z) + d(z,y)$.

Theorem 2.4 [3, 6] : Let X be a complete metrically convex metric space. If, for the set-valued mapping $F : X \to 2^X$, there is a mapping $\psi : P \to \mathbb{R}_+$ satisfying

(i) $D(Fx, Fy) \leq \psi(d(x,y)),\ \forall x, y \in X$, where $D(.,.)$ is a metric on 2^X, defined as $D(A,B) = \sup\{d(x,y) : x \in A,\ y \in B\},\ \forall A, B \in 2^X$;

(ii) $\psi(t) < t,\ \forall t \in \bar{P} \setminus \{0\}$.

Then F has a fixed-point and for any $x_0 \in X,\ x_n \in F(x_{n-1}),\ n \geq 1, \{x_n\}$ converges to a fixed point of F in X.

3. Generalized Set-Valued Implicit Quasi-Variational Inequality Problem

From now onwards, unless otherwise stated, we assume that E is a real uniformly smooth Banach space.

Let $g : E \to E$ be a single-valued mapping and $T, A, S : E \to CB(E)$ be three set-valued mappings. Let $N : E \times E \times E \to E$ be a nonlinear single-valued mapping and

$K: E \to 2^E$ be a set-valued mapping such that for any $x \in E,\ K(x)$ is a nonempty, closed and convex set in E, then we consider the following generalized set-valued implicit quasi-variational inequality problem (for short, GSIQVIP):
Find $x \in E,\ u \in T(x),\ v \in A(x),\ w \in S(x)$ such that $g(x) \in K(x)$ and

$$\langle g(x) + N(u,v,w), J(y - g(x))\rangle \ \geq \ 0,\ \forall y \in K(x). \tag{3.1}$$

We remark that for appropriate choices of mappings g, N, T, A, S, K, and the space E, one can obtain many known classes of variational inequalities from GSIQVIP (3.1), see for example [1, 6, 8, 10-13].
We need the following concepts and results which are needed in the sequel.
Definition 3.1 : Let $T, A, S : E \to CB(E)$. A mapping $N : E \times E \times E \to E$ is said to be:

(i) α-strongly accretive with respect to T, A and S if there exists a constant $\alpha > 0$ such that

$$\langle N(u_1,v_1,w_1) - N(u_2,v_2,w_2), J(x_1 - x_2)\rangle \geq \alpha\|x_1 - x_2\|^2,$$

$$\forall x_1,\ x_2 \in E,\ u_1 \in T(x_1),\ v_1 \in A(x_1),\ w_1 \in S(x_1),\ u_2 \in T(x_2),\ v_2 \in A(x_2),\ w_2 \in S(x_2);$$

(ii) (β, γ, ξ)-mixed Lipschitz continuous if there exist constants $\beta, \gamma, \xi > 0$ such that

$$\|N(x_1,y_1,z_1) - N(x_2,y_2,z_2)\| \leq \beta\|x_1 - x_2\| + \gamma\|y_1 - y_2\| + \xi\|z_1 - z_2\|,$$

$\forall x_1, x_2, y_1, y_2, z_1, z_2 \in E.$

Remark 3.1 : The concept of α-strongly accretiveness with respect to T A and S and (β, γ, ξ)-mixed Lipschitz continuity of mapping $N(\cdot,\cdot,\cdot)$ are more general than the concepts used in [1,6,8,10-13]. If T is μ-$\mathcal{H}$-Lipschitz continuous then $\alpha \leq \beta\mu$.
Assumption 3.1 : For all $x, y, z \in E$, the retraction mapping $R_{K(x)}$ from $E \to K(x)$ satisfies the condition:

$$\|R_{K(x)}(z) - R_{K(y)}(z)\| \leq \nu\|x - y\|, \quad \nu > 0 \text{ is a constant.}$$

4. Main Results

The following lemma, which will be used in the sequel, is an equivalence between the solutions of GSIQVIP (3.1) and a fixed point problem.

Lemma 4.1 : GSIQVIP (3.1) has a solution (x,u,v,w) with $x \in E$, $u \in T(x)$, $v \in A(x)$, $w \in S(x)$, $g(x) \in K(x)$ if and only if the set-valued mapping $F : E \to 2^E$ defined by

$$F(x) = \bigcup_{u\in T(x)} \bigcup_{v\in A(x)} \bigcup_{w\in S(x)} \Big\{x - g(x) + R_{K(x)}[(1-\rho)g(x) - \rho N(u,v,w)]\Big\},\ x \in E, \quad (4.1)$$

has a fixed point $x \in E$, where $\rho > 0$ is a constant.

Proof : (x,u,v,w) with $x \in E$, $u \in T(x)$, $v \in A(x)$, $w \in S(x)$, $g(x) \in K(x)$ is a solution of GSIQVIP (3.1) if and only if (x,u,v,w) satisfies

$$\langle g(x) + N(u,v,w), J(y - g(x))\rangle \geq 0, \quad \forall y \in K(x)$$

$$\Longleftrightarrow \langle g(x) - [(1-\rho)g(x) - \rho N(u,v,w)], J(y-g(x))\rangle \geq 0, \quad \forall y \in K(x), \quad \rho > 0$$

$$\Longleftrightarrow\ g(x) = R_{K(x)}\left[(1-\rho)g(x) - \rho N(u,v,w)\right], \text{ (By Theorem 2.2)}$$

$$\Longleftrightarrow\ x = x - g(x) + R_{K(x)}\left[(1-\rho)g(x) - \rho N(u,v,w)\right]$$

$$\Longleftrightarrow\ x \in \bigcup_{u\in T(x)} \bigcup_{v\in A(x)} \bigcup_{w\in S(x)} \Big[x - g(x) + R_{K(x)}[(1-\rho)g(x) - \rho N(u,v,w)]\Big]$$

$$= F(x).$$

Now, using Lemma 4.1, we prove the following existence theorem for GSIQVIP (3.1).

Theorem 4.1 : Let E be a real uniformly smooth Banach space with $\rho_E(t) \leq ct^2$ for some $c > 0$; let the mapping g be σ-strongly accretive and δ-Lipschitz continuous; let the mappings $T, A, S : E \to CB(E)$ be μ-$\mathcal{H}$-Lipschitz continuous, η-$\mathcal{H}$-Lipschitz continuous and λ-$\mathcal{H}$-Lipschitz continuous, respectively; let the mapping N be α-strongly accretive with respect to T, A and S and (β,γ,ξ)-mixed Lipschitz continuous. If Assumption 3.1 holds and there exists a constant $\rho > 0$ such that

$$\left|\rho - \frac{\alpha - (1-k)\delta}{64c\pi^2 - \delta^2}\right| < \frac{\sqrt{(\alpha - (1-k)\delta)^2 - k(2-k)(64c\pi^2 - \delta^2)}}{64c\pi^2 - \delta^2} \quad (4.2)$$

$$\alpha > (1-k)\delta + \sqrt{k(2-k)(64c\pi^2 - \delta^2)}; \quad \rho\delta < 1 - k; \quad (4.3)$$

$$k := \nu + 2\sqrt{1 - 2\sigma + 64c\delta^2}; \quad 8\sqrt{c}\pi > \delta; \quad \pi := (1+\epsilon)(\beta\mu + \gamma\eta + \xi\lambda). \quad (4.4)$$

Then GSIQVIP (3.1) has a solution (x, u, v, w) with $x \in E$, $u \in T(x)$, $v \in A(x)$, $w \in S(x)$, $g(x) \in K(x)$.

Proof : For applying Lemma 4.1, we need to show that function F defined by (4.1) has a fixed point. Thus, for any $x_1, x_2 \in E$, $p \in F(x_1)$, $q \in F(x_2)$, there exist $u_1 \in T(x_1)$, $v_1 \in A(x_1)$, $w_1 \in S(x_1)$, $u_2 \in T(x_2)$, $v_2 \in A(x_2)$, $w_2 \in S(x_2)$ such that

$$p = x_1 - g(x_1) + R_{K(x_1)}[(1-\rho)g(x_1) - \rho N(u_1, v_1, w_1)]$$

and

$$q = x_2 - g(x_2) + R_{K(x_2)}[(1-\rho)g(x_2) - \rho N(u_2, v_2, w_2)].$$

By using Assumption 3.1, we have

$$\begin{aligned}
\|p-q\| \le\ & \|x_1-x_2-(g(x_1)-g(x_2))\|+\|R_{K(x_1)}[(1-\rho)g(x_1)-\rho N(u_1,v_1,w_1)] \\
& -R_{K(x_2)}[(1-\rho)g(x_2) - \rho N(u_2,v_2,w_2)]\| \\
\le\ & \|x_1 - x_2 - (g(x_1) - g(x_2))\| + \|R_{K(x_1)}[(1-\rho)g(x_1) - \rho N(u_1,v_1,w_1)] \\
& -R_{K(x_2)}[(1-\rho)g(x_1)-\rho N(u_1,v_1,w_1)]\|+\|R_{K(x_2)}[(1-\rho)g(x_1)-\rho N(u_1,v_1,w_1)] \\
& -R_{K(x_2)}[(1-\rho)g(x_2) - \rho N(u_2,v_2,w_2)]\| \\
\le\ & \|x_1 - x_2 - (g(x_1) - g(x_2))\| + \nu\|x_1 - x_2\| + \|[(1-\rho)g(x_1) - \rho N(u_1,v_1,w_1)] \\
& -[(1-\rho)g(x_2) - \rho N(u_2,v_2,w_2)]\| \\
\le\ & 2\|x_1 - x_2 - (g(x_1) - g(x_2))\| + \nu\|x_1 - x_2\| + \rho\|g(x_1) - g(x_2)\| \\
& +\|x_1 - x_2 - \rho[N(u_1,v_1,w_1) - N(u_2,v_2,w_2)]\|. \qquad (4.5)
\end{aligned}$$

Since g is σ-strongly accretive and δ-Lipschitz continuous, using Lemma 2.1, we have

$$\begin{aligned}
\|x_1-x_2-(g(x_1)-g(x_2))\|^2 \le\ & \|x_1-x_2\|^2-2\langle g(x_1)-g(x_2), J(x_1-x_2-(g(x_1)-g(x_2)))\rangle \\
\le\ & \|x_1 - x_2\|^2 - 2\langle g(x_1) - g(x_2), J(x_1 - x_2)\rangle \\
& - 2\langle g(x_1) - g(x_2), J(x_1 - x_2 - (g(x_1) - g(x_2))) - J(x_1 - x_2)\rangle \\
\le\ & \|x_1 - x_2\|^2 - 2\sigma\|x_1 - x_2\| + 64c\delta^2\|x_1 - x_2\|^2 \\
\le\ & (1 - 2\sigma + 64c\delta^2)\|x_1 - x_2\|^2. \qquad (4.6)
\end{aligned}$$

Further, N is (β, γ, ξ)-mixed Lipschitz continuous and T, A, S are μ-$\mathcal{H}$-Lipschitz continuous, η-$\mathcal{H}$-Lipschitz continuous and λ-$\mathcal{H}$-Lipschitz continuous, respectively, we have

$$\begin{aligned}
&\|N(u_1,v_1,w_1)-N(u_2,v_2,w_2)\| \leq \beta\|u_1-u_2\|+\gamma\|v_1-v_2\|+\xi\|w_1-w_2\| \\
&\quad \leq (1+\epsilon)(\beta\mathcal{H}(T(x_1),T(x_2))+\gamma\mathcal{H}(A(x_1),A(x_2))+\xi\mathcal{H}(S(x_1),S(x_2))) \\
&\quad \leq \ (1+\epsilon)(\beta\mu+\gamma\eta+\xi\lambda)\|x_1-x_2\|. \qquad (4.7)
\end{aligned}$$

Furthermore, N is α-strongly accretive with respect to T, A and S, then by using Lemma 2.1 and (4.7), we have

$$\begin{aligned}
&\|x_1-x_2-\rho[N(u_1,v_1,w_1)-N(u_2,v_2,w_2)]\|^2 \\
&\leq \ \|x_1-x_2\|^2-2\rho\langle N(u_1,v_1,w_1)-N(u_2,v_2,w_2), J(x_1-x_2-\rho[N(u_1,v_1,w_1)-N(u_2,v_2,w_2)])\rangle \\
&\leq \ \|x_1-x_2\|^2-2\rho\langle N(u_1,v_1,w_1)-N(u_2,v_2,w_2), J(x_1-x_2)\rangle \\
&\quad -2\rho\langle N(u_1,v_1,w_1)-N(u_2,v_2,w-2), J(x_1-x_2-\rho[N(u_1,v_1,w_1)-N(u_2,v_2,w_2)]-J(x_1-x_2)\rangle \\
&\leq \ [1-2\rho\alpha+64c\rho^2(1+\epsilon)^2(\beta\mu+\gamma\eta+\xi\lambda)^2] \ \|x_1-x_2\|^2. \qquad (4.8)
\end{aligned}$$

Hence, from (4.5), (4.6) and (4.8), we get

$$\begin{aligned}
D\left(F(x_1),F(x_2)\right) &\leq [2\sqrt{1-2\sigma+64c\delta^2}+\nu+\rho\delta \\
&\qquad +\sqrt{1-2\rho\alpha+64c\rho^2(1+\epsilon)^2(\beta\mu+\gamma\eta+\xi\lambda)^2}\ ]\ \|x_1-x_2\| \\
&\leq \ \psi(\|x_1-x_2\|), \qquad (4.9)
\end{aligned}$$

where $\psi(t)=\theta t,\ \theta=k+\rho\delta+l(\rho);\ \ t=\|x_1-x_2\|;\ \ k=\nu+2\sqrt{1-2\sigma+64c\delta^2}$ and $l(\rho)=\sqrt{1-2\rho\alpha+64c\rho^2(1+\epsilon)^2(\beta\mu+\gamma\eta+\xi\lambda)^2}$.

Now, we show that $\theta<1$. It is clear that $l(\rho)$ assumes its minimum value for $\bar{\rho}=\dfrac{\alpha}{64c(1+\epsilon)^2(\beta\mu+\gamma\eta+\xi\lambda)^2}$ with $l(\bar{\rho})=\sqrt{1-\dfrac{\alpha}{64c(1+\epsilon)^2(\beta\mu+\gamma\eta+\xi\lambda)^2}}$. For $\rho=\bar{\rho}$, $k+\rho\delta+l(\rho)<1$ implies $\rho\delta<1-k$. Thus, it follows that $\theta<1$ for all ρ satisfying (4.2)-(4.4). Since Banach space is a metrically convex metric space and $\theta<1$, then $\psi(t)<t$ for each $t\in[0,\infty)$, and hence by Theorem 2.4, F has a fixed point $x\in E$. This completes the proof with Lemma 4.1.

Remark 4.1 : It is clear that $\alpha\leq\beta\mu;\ \sigma\leq\delta$. Further, $\theta<1$ and conditions (4.2)-(4.4) of Theorem 4.1 hold for some suitable values of constants, for example $(\alpha=3,\ \beta=4,\ \gamma=\xi=1,\ \mu=\eta=\lambda=\delta=1,\ \sigma=0.5,\ \epsilon=\nu=\rho=0.1)$.

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